OPERATIONAL CONSIDERATIONS IN PURCHASING MANAGEMENT

by

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Dedicated to my family and teachers.

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This dissertation analyzes three important problems in purchasing operations. Below, we briefly summarize each problem:

In the first chapter, we focus on designing optimal descending mechanisms for constrained procurement. Descending mechanisms for procurement (or, ascending mechanisms for selling) have been well-recognized for their simplicity from the viewpoint of bidders – they require less bidder sophistication as compared to sealed-bid mechanisms. We consider procurement under each of two types of constraints: (1) *Individual/Group Capacities*: limitations on the amounts that can be sourced from individual and/or subsets of suppliers, and (2) *Business Rules*: lower and upper bounds on the number of suppliers to source from, and on the amount that can be sourced from any single supplier. We analyze two procurement problems, one that incorporates individual/group capacities and another that incorporates business rules. In each problem, we consider a buyer who wants to procure a fixed quantity of a product from a set of suppliers, where each supplier is endowed with a privately-known constant marginal cost. The buyer's objective is to minimize her total expected procurement cost. For both problems, we present descending auction mechanisms that are optimal mechanisms. We then show that these two problems belong to a larger class of mechanism design problems with constraints specified by polymatroids, for which we prove that optimal mechanisms can be implemented as descending mechanisms.

In the second chapter, we propose a modification of the well-known Vickrey-Clarke-Groves (VCG) mechanism and investigate its optimality in the context of procurement. It is well-known that the VCG mechanism is optimal for a buyer procuring one unit from a set of symmetric suppliers. For procuring a unit from asymmetric suppliers, Myerson's optimal mechanism can be interpreted as a transformation of the VCG mechanism – both in terms of its allocation and payment – using the virtual cost function. For a more general setting in which multiple units need to be procured from asymmetric suppliers under an arbitrary set of feasibility constraints, we analyze the same transformation of the VCG mechanism. We show that this mechanism is optimal if the feasible region is a *polymatroid*. We also present an example of a non-polymatroidal feasible region for which this mechanism is sub-optimal.

The third chapter considers governmental procurement of food crops from farmers in developing countries and examines the impact of several operational features on the selling behavior of the farmers. Among the various governmental schemes that support agriculture, support prices have been adopted by many developing countries. A support price for an agricultural crop is a guaranteed price at which a governmental entity agrees to purchase that crop from farmers. Despite this surety, the surprising practice of "distressed" selling has been widely observed in practice: Farmers sell a significant portion of their crops to outside agents at prices much *lower* than the support price. We build a tractable stochastic dynamic programming model that captures the salient features of the ground realities – limited as well as uncertain procurement capacity, high holding costs for the farmers, and lack of affordable credit – that conspire to induce distressed selling and, consequently, a significant loss of welfare of the farmers. Using real data on procurement under a support-price program, we establish the accuracy of our model's prediction on the volume of distressed sales. Finally, we show how our model and its solution can serve as a simple and useful tool for policy-makers to assess the relative impact of the improvements in the main determinants of distressed sales.

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CHAPTER 1

INTRODUCTION

We begin with describing the problems studied in Chapters 2–4, discuss the relevant literature and summarize our contributions.

In Chapter 2, we study the design of optimal descending mechanisms for procurement under several practical operational constraints. It is commonly acknowledged that, in addition to cost, operational considerations play an important role in procurement; see e.g., Hohner et al. (2003), Mochari (2004), Elmaghraby (2007), and Wyld (2011). Typical examples of constrained procurement include:

1. *Individual/Group Capacities*: Suppliers typically have finite production capacities that limit the number of units they can supply to the buyer. In addition, buyers often impose upper bounds on the total amounts procured from different subsets of suppliers, to avoid overdependence on the suppliers from a specific subset, or to ensure a sufficient amount of supplier diversity. For instance, a recent Financial Times report (Marsh 2011) names several companies in the engineering sector that have built networks of suppliers in multiple countries to insulate themselves from disruptive events like earthquakes. We provide two more real-world instances of the use of such constraints in procurement: (a) AT&T's global supplier diversity program (AT&T 2016) targets up to 15% of their total spend from minority businesses and up to 1.5% from service-disabled-veteran businesses. (b) The Hackett Group – an Atlanta-based business advisory firm – recommends that companies give favored suppliers 80 percent of their business and backup vendors 20 percent (Mochari 2004).

2. *Business Rules*: For strategic reasons, buying firms often include the following constraints while sourcing:

    - A lower bound on the number of suppliers to source from and an upper bound on the quantity that can be sourced from any selected supplier. This avoids dependence on too small a supply base.

- An upper bound on the number of suppliers to source from and a lower bound on the quantity that can be sourced from any selected supplier. This avoids the high administrative burden of working with too large a supply base.

Hohner et al. (2003) describe the use of these business rules in the procurement of raw material at Mars, Inc., a producer of many top brands of confectionery, pet food, and rice.

We study two procurement problems – one that has individual/group capacities and another that has business rules as constraints. In both problems, there is a buyer who wants to procure a fixed, aggregate quantity of a product. There are many suppliers, whose production costs are linear functions of production quantity. The unit cost of each supplier is private information, while the distributions of the costs of the suppliers are independent and common knowledge. Examples of papers that make all of these assumptions for all or a significant part of their analysis are Chaturvedi et al. (2013), Wan et al. (2012), Wan and Beil (2009), and Kostamis et al. (2009). The objective of the buyer is to minimize her total expected procurement cost. For both problems, we design descending mechanisms that are optimal. Our goal of designing descending mechanisms is motivated by the observation that such mechanisms require much less bidder sophistication (as elaborated below); see, e.g., Ausubel (2004), Ausubel and Cramton (2006) and Duenyas et al. (2013). In each of these papers, the authors have advocated the use of descending mechanisms in place of sealed-bid mechanisms.

In proving the optimality of the descending mechanisms for the above two procurement problems (namely, Individual/Group Capacities and Business Rules), we analyze a larger class of mechanism design problems that incorporate constraints specified by a polymatroid and provide a descending mechanism that is optimal for a general problem in this class. Finally, we also consider a mechanism design problem for procurement under concave production costs and constraints that are specified by a symmetric polymatroid, and present a descending mechanism that is optimal for this problem.

The descending (auctions) mechanisms proposed here consist of buyer-controlled, continuously decreasing price meters, one for each supplier. Each meter is initialized at a certain unit price and is visible only to the supplier to whom it is assigned. This initial price could be different for different price meters. A supplier can leave the auction at any time by pressing an 'off' button. Until this event, the buyer can award business to the supplier at any time instant, for a unit price equal to the prevailing value of his price meter. The supplier could be awarded business at multiple time instants, before his exit. Once he presses his off button, the supplier is not allowed to participate further in the auction and does not receive any additional business. Thus, the only decision for a supplier is to choose when to exit the auction (i.e., press the off button).

Trivial Identification of the Dominant Strategy: An appealing feature of the above auction is that it is a dominant strategy for each supplier to exit the auction when his price meter hits his privately-known unit cost. This is trivial to see: On the one hand, staying in the auction after the meter has reached his unit cost can only reduce the supplier's profit, because any subsequent awards that are made to him will be at prices that are below his unit cost. On the other hand, by leaving before the meter hits his unit cost, the supplier foregoes the potential to receive additional profitable awards. An additional attractive feature in practice is that the buyer is able to control the duration of the auction, by controlling the speeds of the price meters.

Our work joins the growing stream of research in the Operations Management area on procurement, using analytical or behavioral methods from economics. We refer the reader to the survey by Elmaghraby (2007), and the tutorials by Beil (2010) and Katok (2011).

In Chapter 3, we investigate the optimality of a modified VCG mechanism. The theory of mechanism design in procurement, to a large extent, is concerned with the design of two classes of mechanisms: (1) Optimal mechanisms are the ones that maximize the expected surplus of the buyer. (2) Efficient mechanisms are those that maximize the expected social surplus, which includes both the buyer's expected surplus and the total expected surplus of the suppliers. Consider the simple setting of a buyer who wants to procure one unit of a product from a set of symmetric suppliers, each with a privately-known cost of production. It is well known that the VCG mechanism is both efficient and optimal in this setting; see e.g. Krishna (2002). In this chapter, we focus on the following general, constrained procurement setting:

A buyer wants to procure multiple units of a product from either a set $\mathcal{N} = \{1, 2, \ldots, N\}$ of asymmetric suppliers, each with a privately-known unit-cost, or an outside option with unit-cost R. We denote the outside option as supplier $N + 1$. Each supplier $i \in \mathcal{N}$ has a privately-known unit-cost c_i, which is a realization of a random variable with a continuous cumulative distribution function F_i and a continuous probability density function f_i on the support $[\underline{c}_i, \overline{c}_i]$. The distributions of the costs of the suppliers are independent and common knowledge. The allocations made to the suppliers are subject to a set, denoted by FEAS, of feasibility constraints (see Bichler et al. (2006) for several practically important feasibility constraints). The goal is to find an optimal mechanism for the buyer.

Under this procurement setting, we define a transformation of the VCG mechanism which we refer to as the “virtual” VCG mechanism (VVCG). VVCG-like mechanisms have precedence in the literature. For example, Chapter 3 of Hartline (2013) considers a forward setting in which a seller sells multiple units of a product to a set of N bidders with a restriction that each bidder receives an allocation of either 0 or 1 unit. For this setting, the author proposes an optimal sealed-bid mechanism, which is similar to the VVCG mechanism in spirit. Another example is Duenyas et al. (2013) who consider a buyer procuring an endogenously determined quantity of a product from a set of N suppliers. For this setting, the authors propose an optimal descending mechanism in which the use of virtual cost functions in designing the allocations and payments given to the suppliers is similar to that used in the VVCG mechanism.

VVCG is a natural extension of the VCG mechanism and uses virtual cost functions that play a prominent role in Myerson (1981). We know that VCG is an efficient mechanism. We also know from Duenyas et al. (2013) and Hartline (2013) that VVCG is optimal for the specific settings they study. Given this, it is interesting to understand if VVCG is always an optimal mechanism. This chapter provides a definitive answer to this question by demonstrating VVCG’s optimality for a large variety of settings (i.e., polymatroid feasible allocation sets) and by showing that VVCG fails to be optimal in other settings.

In Chapter 4, we study the procurement of food crops in developing countries. In most developing economies, the agricultural sector contributes a significant share to the GDP, e.g., about 20% in Bangladesh, 12% in Philippines, 18% in India, 15% in Indonesia, and 20% in Vietnam (Food and Agriculture Organization of the UN 2015, World Bank 2015). A significant percentage of the poor in developing countries live in rural areas and depend on agriculture for their livelihood. Among these, perhaps the most vulnerable group is that of the small and marginal farmers. With limited land holdings, these farmers continually suffer from rising production costs, market-price fluctuations of their crops, lack of access to affordable credit, and a poor logistics infrastructure to transport their crops to urban markets. Given the central role these farmers play in ensuring food security, it is no surprise that policy makers in developing countries offer a variety of schemes that are aimed at improving their welfare.

One such governmental intervention is through support prices. A support price for an agricultural crop is a guaranteed and attractive price at which the government agrees to directly purchase that crop from farmers. Support prices for major crops have been adopted by many developing countries, including Bangladesh, Brazil, China, India, Pakistan, Philippines, Thailand, and Turkey. The food grains procured under a support-price program are then made available to the poor at an affordable price. Thus, the government's motivation behind offering a support price for a crop is threefold: incentivizing production of the crop, protecting the farmers against lower market prices, and feeding the economically-weak segment of the population. The following quote from the Ministry of Finance, India (2010) succinctly states these goals:

> [A support-price program] for agricultural produce seeks to ensure remunerative prices to growers for their produce with a view to encourage higher investment and production as well as safeguarding the interests of consumers by making available supplies at reasonable prices. The price policy also seeks to evolve a balanced and integrated price structure in the perspective of the overall needs of the economy.

Despite the good intentions behind price support, a variety of operational challenges in the procurement (by the government) under this program lead to farmers often being forced to sell their produce to outside agents at prices that are much lower than the support price. We now examine the reasons behind this practice, which is commonly referred to as *distressed selling*[1].

The operational details of governmental procurement under agricultural price support naturally differ across countries. A massive scale of procurement and a better-organized procurement process – relative to other developing countries – make India a good case for understanding distressed selling.

The Indian government offers support prices for most major crops, including wheat, rice, and lentils. The government implements this for a crop by announcing a support price at the start of its growing season and subsequently opening procurement centers across the country. Each Government Procurement Center (GPC) covers a specific geographical area; all the farmers in this area are assigned to the corresponding GPC. After harvest, the farmers transport their produce to their GPC to complete the sales and receive full payment.

A major obstacle in this process, however, is the limited as well as the uncertain capacity at a GPC. Even with a large number of GPCs across the country, the massive population of the farmers implies that multiple villages are assigned to the same GPC. Although the government guarantees procurement from all the farmers who are willing to sell their harvest at the GPC, the processing rate of the farmers' transactions at the GPC is slow, as it involves, among other steps, grading, sorting, and weighing of the grains (The Indian Express 2014, The Hindu 2014a). Thus, the government's procurement capacity over the selling season is limited. There is capacity uncertainty too; the following examples are illustrative: (i) The

[1] See `https://www.youtube.com/watch?v=U2lr5rBTpaU` for a 1-minute news clip on distressed selling.

grains procured by the GPC are transported to central warehouses and, from there, to fair-price shops across the country, where the economically-backward population can access them. If the outflow from a central warehouse is hampered, say due to inclement weather, then the inflow to this warehouse from its associated GPCs is reduced, ultimately slowing down procurement activities at the GPCs. (ii) A stock-out of the bags – that are used to store the procured grains – at the GPC can suddenly close down procurement operations (Economic and Political Weekly 2011).

While waiting to sell his produce at the GPC, the farmer is responsible for the temporary holding of his grains near the GPC; the storage infrastructure, however, is both inadequate and expensive in the commercial centers where the GPCs are typically located (The Hindu 2014b). Worsening this situation is the fact that farmers are poor and debt-ridden from the loans to finance their production costs, often at high interest rates from opportunistic lenders (Sahu et al. 2004). Thus, farmers are in an urgent need of cash during the selling season and waiting to sell their crops at the GPC is costly. Opportunistic market agents (or *outside agents*) exploit this situation and buy from the farmers at prices much lower than the support price (The Economic Times 2015, The Financial Express 2015, The Times of India 2013, Business Standard 2012a,b), resulting in distressed selling. To illustrate the magnitude of this phenomenon, Table 1.1 shows the percentage share of the government's procurement in some major wheat-producing states in India in the 2011–12 and 2012–13 selling seasons (Directorate of Economics and Statistics 2013, 2014).

The practice of distressed selling has been reported in several other developing countries as well; e.g., Bangladesh (Mandal 2010), Pakistan (Salman 2011), Tanzania (Knudsen and Nash 1990), and Ethiopia (Minot and Rashid 2013).

Table 1.1. Percentage share of government purchase in some wheat producing states in India; production and procurement quantities are specified in million tonnes (1 tonne = 1000 kg).

State	2011–12			2012–13		
	Total Production	Total Procurement	Percentage Procured	Total Production	Total Procurement	Percentage Procured
Punjab	17.28	10.96	63.43%	16.59	12.83	77.34%
Haryana	12.69	6.93	54.61%	11.12	8.67	77.97%
Rajasthan	9.32	1.3	13.95%	9.28	1.96	21.12%
Bihar	4.73	0.56	11.84%	5.36	0.77	14.37%
Gujarat	4.07	0.11	2.70%	2.94	0.16	5.44%

CHAPTER 2

OPTIMAL DESCENDING MECHANISMS FOR CONSTRAINED PROCUREMENT[1]

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[1]Portions of this chapter are reprinted with permission: Shivam Gupta, Wei Chen, Milind Dawande, Ganesh Janakiraman. "Optimal Descending Mechanisms for Constrained Procurement." *Production and Operations Management*, **24**(12), 1955–1965, 2015.

This chapter is organized as follows: In Sections 2.1 and 2.2, we formally define our two procurement problems – one that has individual/group capacities and another that has business rules as constraints, and propose optimal descending mechanisms for both these problems. In Section 2.3, we construct an optimal descending mechanism for procurement under polymatroidal feasibility constraints. Section 2.4 presents an extension considering suppliers with concave production costs. Finally, we conclude our chapter in Section 2.5 and discuss future research directions.

2.1 Procurement Under Individual/Group Capacities

Our first procurement problem, referred to as M-CGP, is defined as follows: Consider a buyer and a set $\mathcal{N}$ of suppliers, where $|\mathcal{N}| = N$. Each supplier $i \in \mathcal{N}$ has a private unit cost c_i, which is a realization of a random variable with a continuous cumulative distribution function F_i and a continuous probability density function f_i on the support $[\underline{c}_i, \bar{c}_i]$. There are M *disjoint* groups of suppliers, with index sets denoted by $G_k \subseteq \mathcal{N}, k = 1, 2, \ldots, M$. Thus, $\cup_{k=1}^{M} G_k \subseteq \mathcal{N}$ and $G_k \cap G_{k'} = \emptyset$ for $k \neq k'$. Let $\mathcal{M} = \{1, 2, \ldots, M\}$ and $G^0 = \mathcal{N} \setminus \cup_{k=1}^{M} G_k$ be the collection of suppliers who do not belong to any group. The buyer wants to procure a fixed quantity Q of a product. The objective of the buyer is to minimize the total expected procurement cost, subject to the usual incentive compatibility and individual rationality constraints, and the following constraints:

- (**Individual Supplier Capacities**) The total amount that can be procured from Supplier i can be at most his production capacity Γ_i, $i \in \mathcal{N}$.
- (**Group Procurement Constraints**) The total amount that can be procured from the suppliers in Group G_k can be at most $\eta_k, k \in \mathcal{M}$.

Without loss of generality, we assume that Q, $\Gamma_i, i \in \mathcal{N}$, and $\eta_k, k \in \mathcal{M}$, are positive integers. It is straightforward to verify that the constraints admit a feasible solution if and only if

$$Q \leq \sum_{i \in G^0} \Gamma_i + \sum_{k=1}^{M} \min \left\{ \eta_k, \sum_{i \in G_k} \Gamma_i \right\}. \tag{2.1}$$

Therefore, we will assume this condition throughout our discussion of M-CGP.

Before we formally describe our descending mechanism for this problem, we illustrate it on a simple numerical example. At the end of this illustration, we also comment on the connection of our mechanism with that in Ausubel (2004).

2.1.1 Illustrative Example

Consider a buyer and a set of 4 suppliers. Each supplier has an independent and identically distributed marginal cost in an interval $[\underline{c}, \bar{c}]$ with a cumulative distribution function F. The buyer wants to procure $Q = 10$ units. For expositional convenience, we refer to the first supplier as Supplier A, the second as Supplier X, the third as Supplier B, and the fourth as Supplier Y. The individual production capacities of these suppliers are as follows: $\Gamma_A = 4, \Gamma_X = 2, \Gamma_B = 5$, and $\Gamma_Y = 4$. There are two groups of suppliers (thus, $\mathcal{M} = \{1, 2\}$): $G_1 = \{A, B\}$ and $G_2 = \{X, Y\}$. The total amount that can be procured from the suppliers in G_1 (resp., G_2) is at most $\eta_1 = 7$ (resp., $\eta_2 = 4$).

To derive intuition, let us first consider the deterministic case, where the buyer knows the costs of the suppliers. Without loss of generality, let $c_Y > c_B > c_A > c_X$. The optimal allocation in this case is easy to understand: In order to minimize the total procurement cost, the buyer awards the maximum feasible amount to Supplier X. This supplier has a production capacity of 2 units and belongs to Group 2, which has a group capacity of 4 units. Therefore, Supplier X is awarded 2 units. The next cheapest supplier, i.e., Supplier A, is now considered. This supplier has a production capacity of 4 units and his group capacity is 7 units; therefore, he is awarded 4 units. The next cheapest supplier, i.e., Supplier B, has a production capacity of 5 and belongs to Group 1, which has a group capacity of 7 units. Since Supplier A, who also belongs to Group 1, has already been awarded 4 units, Supplier B gets an award of only 3 units. Finally, Supplier Y receives 1 unit. To summarize, Supplier X receives 2 units, A receives 4 units, B receives 3 units and Y receives 1 unit.

We now return to the case when the buyer does not know the costs of the suppliers. The allocations made by our descending mechanism are discussed step-by-step below. For simplicity of illustration, let us assume that the costs of the suppliers (which are unknown to the buyer) are in the same order as above, i.e., $c_Y > c_B > c_A > c_X$. It will turn out that the final allocations to the suppliers by the mechanism are the same as those in the deterministic setting discussed above. The descending auction in this example works as follows: There is a common price meter for all suppliers. Let this meter start at price $P(0) = \bar{c}$ and decrease continuously, e.g., following the equation $P(t) = \bar{c} - t$. Recall from Chapter 1 that it is a dominant strategy for each supplier to exit when the price meter hits his unit cost. As a consequence, the suppliers leave in the order of their unit costs, starting with the supplier with the highest unit cost.

At time $t = 0$, the buyer makes the following calculations: (i) the total number of units needed is 10, (ii) the total residual capacity of the remaining suppliers is 11, and (iii) the

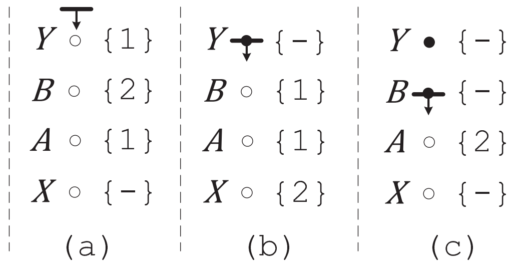

Figure 2.1. An illustration of our descending mechanism for M-CGP. Shaded circles represent suppliers who have exited the auction, whereas unshaded circles represent suppliers who continue to participate. The amounts $\{\cdot\}$ indicate the incremental allocation given to a supplier at the corresponding time instant. The total amount allocated to a supplier is the sum of these incremental allocations.

total residual capacity of the suppliers excluding Supplier A (resp., Supplier B, Supplier X, Supplier Y) is 9 (resp., 8, 11, 9). If the price meter is reduced and, as a consequence, any one of the suppliers A, B, or Y leaves, then no feasible allocation among the remaining suppliers will satisfy the total requirement of the buyer. Therefore, the buyer awards 1 (= 10-9) unit to Supplier A, 2 (= 10-2) units to Supplier B, and 1 (= 10-9) unit to Supplier Y. Each of these awards is made at a unit price equal to the current price meter (which is $P(0)$).

The meter now reduces until exactly one supplier has left the auction (Supplier Y, in this example). At this time instant, again the buyer does the following calculations: (i) the total number of additional units needed is 6, (2) the total residual capacity of the remaining suppliers is 6, (3) the total residual capacity of the remaining suppliers excluding Supplier A (resp., supplier B, Supplier X) is 5 (resp., 5, 4). Using the same logic as above, the buyer awards 1 unit to Supplier A, 1 unit to Supplier B, and 2 units to Supplier X; again, each of these awards is made at a unit price equal to the current price meter.

The meter continues to decrease until exactly one more supplier has left the auction (Supplier B, in our case). At this time instant, the total number of units needed is 2, the residual capacity of Supplier A is 2, and the residual capacity of Supplier X is 0. Therefore, the buyer awards 2 units to Supplier A at a unit price equal to the current price meter. At this point, the buyer has procured the required amount and, therefore, the auction ends.

Figure 2.1 gives a pictorial representation of the awards made at each of the above three time instances. To summarize, Supplier Y receives a cumulative allocation of 1 unit, B receives 3 units, A receives 4 units, and X receives 2 units. The optimality of this mechanism is formally established in Theorem 1 (Section 2.1.2).

Connection of Our Descending Mechanism to that in Ausubel (2004): Ausubel (2004) considers a forward setting in which the auctioneer sells multiple (homogeneous) units. Each bidder has different marginal valuations for different units and is allowed to buy multiple units. There are no set-based constraints, akin to our group procurement constraints. For this problem, Ausubel designs an ascending auction that is efficient. Ausubel's auction, for the reverse setting, solves M-CGP *in the absence of the group procurement constraints for the special case when the costs of the suppliers are from the same distribution.* Thus, our descending mechanism and Ausubel's auction share certain similarities, e.g., in our descending mechanism, at any instant of time when a supplier receives an award, the per-unit price paid to the supplier for this award is equal to the value of his price meter at that time instant. In Ausubel (2004), this is referred to as "clinching". Our problem is more general in that we consider set-based constraints, while Ausubel's problem is more general in that he allows for non-constant marginal costs. Finally, we reiterate that our goal is to obtain an *optimal* mechanism, whereas the goal in Ausubel (2004) is to obtain an *efficient* mechanism: An optimal mechanism maximizes the buyer's expected surplus, while an efficient mechanism maximizes the sum of the buyer's surplus and the total surplus of all the suppliers.

2.1.2 Descending Mechanism

At any time during the auction, we define the following variables: (1) $\hat{\mathcal{N}}$ is the set of suppliers remaining in the auction, (2) $NEED$ is the number of units that remains to be procured, (3) RC_i is an upper bound on the number of additional units that can be procured from Supplier i, (4) RCG_l is an upper bound on the number of additional units that can be procured from the suppliers in Group G_l, (5) $TRCwS_i$ is the total residual capacity of the remaining suppliers excluding Supplier i, and (6) $ALLOC_i$ is the cumulative award given to Supplier i.

Define $\psi_i(c) = c + F_i(c)/f_i(c)$ for any $i \in \mathcal{N}$ and $c \in [\underline{c}_i, \bar{c}_i]$, and assume that $\psi_i(c)$ is strictly increasing in c. Our descending mechanism for problem M-CGP is as follows:

- **Step 1**: (**Supplier-specific Price Meters**) At any time $t \geq 0$, Supplier i observes a continuously decreasing price meter $P_i(t)$, defined as follows:

$$P_i(t) = \psi_i^{-1}\left[\min\left\{\psi_i(\bar{c}_i), \max_j \psi_j(\bar{c}_j) - rt\right\}\right], \tag{2.2}$$

  where r is the rate at which the price is reduced and is a parameter chosen by the buyer. We note a few observations about the expression (2.2) for the price meters:

(a) The price meter of Supplier i is initialized at $P_i(0) = \bar{c}_i$. (b) When the suppliers are symmetric, a common price meter suffices. (c) For Supplier i, the meter stays constant at $\bar{c}_i$ until a certain time $\bar{t}_i$ (defined by $\max_{j\in\mathcal{N}} \psi_j(\bar{c}_j) - r\bar{t}_i = \psi_i(\bar{c}_i)$). After that time, his price meter follows the equation $\psi_i(P_i(t)) = \max_{j\in\mathcal{N}} \psi_j(\bar{c}_j) - rt$, which is independent of i. Thus, for $t \geq \max_s \bar{t}_s$, the price meters decrease such that $\psi_i(P_i(t)) = \psi_j(P_j(t)), \forall i, j$. If the price meter $P_i(t)$ of Supplier i hits the lower bound $\underline{c}_i$ of his private cost support at any time instant $t = t^0$, then his price meter is assumed to stay at $\underline{c}_i$ for $t \geq t^0$. These observations, along with the fact that each supplier i exits when $P_i(t) = c_i$, imply that the suppliers exit the auction in the decreasing order of the virtual costs, $\psi_i(c_i)$.

- **Step 2**: (**Initialization**) At the start of the auction, the buyer sets the following: (a) $NEED = Q$. (b) $\hat{\mathcal{N}} = \mathcal{N}$. (c) For all $l \in \mathcal{M}$, $RCG_l = \eta_l$. (d) For all $i \in \hat{\mathcal{N}}$, $\hat{P}_i = P_i(0)$, $ALLOC_i = 0$, $RC_i = \Gamma_i$ and $TRCwS_i$ is as follows:

$$TRCwS_i = \begin{cases} \sum_{j\in G^0} \Gamma_j - \Gamma_i + \sum_{l\in\mathcal{M}} \min\left\{\eta_l, \sum_{j\in G_l} \Gamma_j\right\}, & \text{if } i \in G^0, \\ \\ \sum_{j\in G^0} \Gamma_j + \min\left\{\eta_s, \sum_{j\in G_s\setminus\{i\}} \Gamma_j\right\} + \\ \sum_{l\in\mathcal{M}, l\neq s} \min\left\{\eta_l, \sum_{j\in G_l} \Gamma_j\right\} & \text{if } i \in G_s, s \in \mathcal{M}. \end{cases}$$

  The buyer implements Step 3.

- **Step 3**: (**Possible Awards**) The buyer awards $\max\{0, NEED - TRCwS_i\}$ to each Supplier $i \in \hat{\mathcal{N}}$ at unit price equal to $\hat{P}_i$. After these awards are made, the buyer updates $NEED$; $ALLOC_i$ and RC_i for all $i \in \hat{\mathcal{N}}$; and RCG_l for all $l \in \mathcal{M}$ in the following sequence:

  - $ALLOC_i = ALLOC_i + \max\{0, NEED - TRCwS_i\}$
  - $NEED = Q - \sum_{i\in\mathcal{N}} ALLOC_i$,
  - $RC_i = \Gamma_i - ALLOC_i$,
  - $RCG_l = \eta_l - \sum_{j\in G_l} ALLOC_j$.

  If $NEED = 0$, then the auction ends. Otherwise, the buyer implements Step 4.

- **Step 4**: (**Elimination**) The buyer reduces the price meters until a supplier, denoted by $\hat{i}$, leaves the auction. At this time instant, the buyer sets $\hat{\mathcal{N}} = \hat{\mathcal{N}} \setminus \{\hat{i}\}$ and $\hat{P}_i$

equal to the current price meter of Supplier i for all $i \in \hat{\mathcal{N}}$. Let $\hat{G}^0 = G^0 \cap \hat{\mathcal{N}}$ and $\hat{G}_l = G_l \cap \hat{\mathcal{N}}$ for all $l \in \mathcal{M}$. The buyer sets $TRCwS_i$ for all $i \in \hat{\mathcal{N}}$ as follows:

$$TRCwS_i = \begin{cases} \sum_{j \in \hat{G}^0} RC_j - RC_i + \sum_{l \in \mathcal{M}} \min\left\{RCG_l, \sum_{j \in \hat{G}_l} RC_j\right\}, & i \in \hat{G}^0, \\ \\ \sum_{j \in \hat{G}^0} RC_j + \min\left\{RCG_s, \sum_{j \in \hat{G}_s} RC_j - RC_i\right\} + \\ \sum_{l \in \mathcal{M}, l \neq s} \min\left\{RCG_l, \sum_{j \in \hat{G}_l} RC_j\right\}, & i \in \hat{G}_s, s \in \mathcal{M}. \end{cases}$$

The buyer returns to Step 3.

Theorem 1. *The descending mechanism above is an optimal solution to M-CGP.*

The proof of Theorem 1 is provided in Section 2.3.2.

2.2 Procurement Under Business Rules

Our second procurement problem, referred to as M-BR, is defined as follows: There is a buyer and a set $\mathcal{N}$ of suppliers. Each supplier $i \in \mathcal{N}$ has a private unit cost c_i, which is a realization of a random variable with a continuous cumulative distribution function F_i and a continuous probability density function f_i on the support $[\underline{c}_i, \bar{c}_i]$. The buyer wants to procure a fixed quantity Q of a product, subject to the following business rules:

- The number of winning suppliers must be at least a minimum number, N_L.
- The number of winning suppliers must be at most a maximum number, N_H.
- Each winning supplier must be allocated a minimum fraction, a, of the total business.
- Each winning supplier can be given at most a maximum fraction, b, of the total business.

The buyer's objective is to minimize the total expected procurement cost, subject to the usual incentive compatibility and individual rationality constraints and the above business rules. It is easy to see that, under the above business rules, there exists a feasible allocation if and only if there exists $n \in \{N_L, N_L + 1, \ldots, N_H\}$ such that $(aQ)n \leq Q \leq (bQ)n$. Let L denote the smallest value of n that satisfies this condition.

Independent of our work, Hu et al. (2013) note some comments (in the concluding section of that paper) that are related to our analysis of business rules in this section. For their problem, the authors mention an allocation process for the case when the buyer uses specific percentages to split the total award among multiple suppliers. This allocation process is similar to our descending mechanism in Section 2.2.2.

As in the previous section, we first illustrate our descending mechanism for M-BR on a simple example.

2.2.1 Illustrative Example

There is a buyer and a set of 6 suppliers. Each supplier has an independent and identically distributed marginal cost in an interval $[\underline{c}, \bar{c}]$ with a cumulative distribution function F. The buyer wants to procure 100 units of a product, subject to the following business rules: (1) At least three and at most five suppliers must be selected, and (2) Each selected supplier must be given a minimum of 20 units and a maximum of 50 units of the product.

Let us first examine how the buyer would award the 100 units of business if she knew the unit costs of the suppliers. Without loss of generality, let the costs of Suppliers 1–6 be such that $c_1 < c_2 < c_3 < c_4 < c_5 < c_6$. The optimal allocation in this case is easy to understand and is as follows: Supplier 1 receives an award of 50 units, Supplier 2 receives 30 units, Supplier 3 receives 20 units, and Suppliers 4–6 do not receive any award.

We now return to the case when the buyer does not know the costs of the suppliers. For purpose of illustration, let us assume that the costs of the suppliers (which are unknown to the buyer) are in the same order as above, i.e., $c_1 < c_2 < c_3 < c_4 < c_5 < c_6$. As we will soon see, the final allocations to the suppliers by our mechanism will be the same as that for the deterministic setting above. For this example, our descending auction works as follows: There is a common price meter for all suppliers. Let this meter start at a price $P(0) = \bar{c}$ (Figure 2.2a) and decrease continuously, e.g., following the equation $P(t) = \bar{c} - t$. From our discussion in Chapter 1 and Section 2.1.1, we know that as the price meter reduces, suppliers leave in the order of their unit costs, starting from the supplier with the highest unit cost (Figure 2.2b). The price meter continues to decrease until all but three suppliers have left the auction (Figure 2.2d). At this time instant, the buyer awards each of the remaining three suppliers 20 units at a unit price equal to the current price meter. The meter continues to decrease until all but two suppliers have left the auction (Figure 2.2e). At this time instant, the buyer awards each of the remaining two suppliers 10 units at a unit price equal to the current price meter. The meter continues to decrease until one more supplier has left the auction (Figure 2.2f). At this time instant, the buyer awards the remaining supplier 20 units at a unit price equal to the current price meter. The auction ends. To summarize, Supplier 1

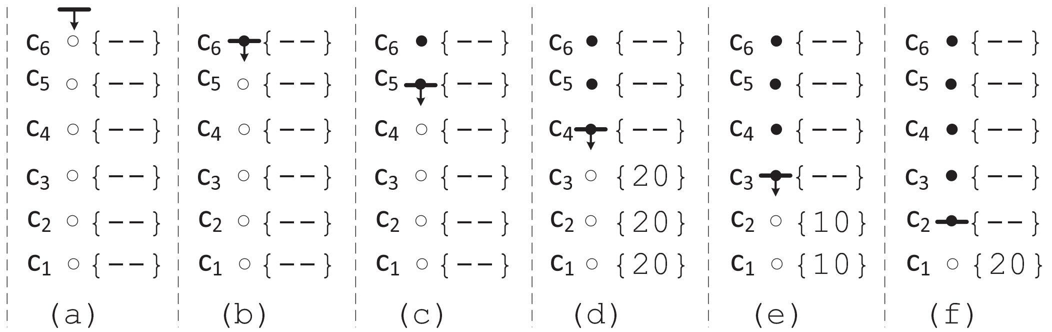

Figure 2.2. An illustration of a descending mechanism for M-BR.

receives a cumulative award of 50 units, Supplier 2 receives 30 units, Supplier 3 receives 20 units, and Suppliers 4–6 do not receive any award. The optimality of this mechanism is formally established in Theorem 2 (Section 2.2.2).

2.2.2 Descending Mechanism

Recall that L is the smallest value of $n \in \{N_L, N_L+1, \ldots, N_H\}$ such that $(aQ)n \leq Q \leq (bQ)n$. Let $K = \lfloor \frac{1-aL}{b-a} \rfloor$ and $\beta = 1 - aL - (b-a)K$. The description of our descending mechanism for problem M-BR is as follows:

- At any time $t \geq 0$, Supplier i observes a continuously decreasing price meter $P_i(t)$, defined as follows:

$$P_i(t) = \psi_i^{-1}\left[\min\left\{\psi_i(\bar{c}_i),\ \max_{j \in \mathcal{N}} \psi_j(\bar{c}_j) - rt\right\}\right], \tag{2.3}$$

  where r is the rate at which the price is reduced and is a parameter chosen by the buyer.

- The buyer continues to reduce the price meters until all but L suppliers have left the auction. At this time instant, the buyer awards each of the remaining L suppliers aQ units at unit prices equal to their respective price meters.

- The buyer continues to reduce the price meters until all but $K+1$ suppliers have left the auction. At this time instant, the buyer awards each of the remaining $K+1$ suppliers βQ units at unit prices equal to their respective price meters.

- The buyer continues to reduce the price meters until one more supplier has left the auction. At this time instant, the buyer awards each of the remaining K suppliers

$[b - a - \beta]Q$ units at unit prices equal to their respective price meters. The auction ends.

Theorem 2. *The descending mechanism above is an optimal solution to M-BR.*

The proof of Theorem 2 is provided in Section 2.3.3.

In the next section, we will show that the descending mechanisms presented above for problems M-CGP and M-BR are special cases of a mechanism that is optimal for a wider class of mechanism design problems. We will define this general mechanism design problem, derive its optimal solution, and show that the mechanisms in Sections 2.1 and 2.2 are special cases of this optimal solution. Finally, we will argue that these mechanisms are optimal for their respective problems, thus establishing Theorems 1 and 2.

2.3 Procurement Under Polymatroid Constraints

Let $f\colon 2^{\mathcal{N}} \to \mathbb{R}_+$ be a non-decreasing and submodular set function with $f(\emptyset) = 0$. That is, we have: (1) $f(\mathcal{S}) \le f(\mathcal{T})$, for all $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{N}$ (non-decreasing), and (2) $f(\mathcal{S} \cup \{s\}) - f(\mathcal{S}) \ge f(\mathcal{T} \cup \{s\}) - f(\mathcal{T})$, for all $\mathcal{S} \subseteq \mathcal{T} \subset \mathcal{N}$, $s \in \mathcal{N} \setminus \mathcal{T}$ (submodular). Let

$$P_f = \left\{(Q_1, Q_2, \ldots, Q_N) \in \mathbb{R}_+^N : \sum_{i \in \mathcal{S}} Q_i \le f(\mathcal{S}),\ \forall\, \mathcal{S} \subseteq \mathcal{N}\right\}$$

be the polymatroid associated with f (Edmonds 1971). Consider the following procurement problem, which we will denote as M-PM(f):

There is a buyer and a set $\mathcal{N}$ of suppliers, where $|\mathcal{N}| = N$. Each supplier $i \in \mathcal{N}$ has a private unit cost c_i. The buyer wants to procure a fixed quantity $f(\mathcal{N})$ with the objective of minimizing her total expected cost, subject to the usual incentive compatibility and individual rationality constraints, and feasibility constraints (governing the allocations given to the suppliers) that are specified by the polymatroid P_f.

Next, we provide an intuitive explanation of the set function f used in the definition of M-PM(f).

- For an arbitrary subset $\mathcal{S}$ of suppliers, the function $f(\mathcal{S})$ represents the maximum allocation that can be feasibly given to the suppliers belonging to the set $\mathcal{S}$. The difference $f(\mathcal{S} \cup \{s\}) - f(\mathcal{S})$ represents the value (that is, the increase in the maximum allocation that can be feasibly given to the suppliers) of adding the supplier s to the subset $\mathcal{S}$.

- The assumption that the set function f is *non-decreasing* means that the value of an additional supplier to any subset of suppliers is non-negative.

- The assumption that the set function f is *submodular* intuitively means that the value of an additional supplier diminishes as the set of suppliers to which the new supplier is added grows.

Examples of papers that consider polymatroid feasibility constraints in an auction framework include Bikhchandani et al. (2011) and Goel et al. (2012). In particular, our descending mechanism in Section 2.3.1 below is more closely related to that of Goel et al. (2012); we comment on this connection immediately after presenting our mechanism. Here, we briefly comment on the intuition behind the descending mechanism: To solve our generalized mechanism design problem, we need to solve an optimization problem over a polymatroid. For this problem, the well-known greedy algorithm of Edmonds (1971) provides an optimal solution and has the following property: *The solution obtained by the greedy algorithm depends only on the order of the virtual-costs of the suppliers.* This is the main property that is exploited in developing a descending mechanism that is optimal for the mechanism design problem.

The remainder of this section is organized as follows: In Section 2.3.1, we define a descending mechanism for M-PM(f), which we will denote as DM($\mathcal{N}$, f) and establish its optimality in Theorem 3. Using this development, we prove Theorems 1 and 2 in Sections 2.3.2 and 2.3.3, respectively. In Section 2.3.4, we consider an extension to Problem M-PM(f), where the total quantity to be procured by the buyer is endogenous, and the buyer obtains a revenue that is linear in quantity. For this problem, we propose a descending mechanism that is optimal.

2.3.1 Descending Mechanism

The descending mechanism DM($\mathcal{N}$, f) works as follows:

- **Step 1**: (**Supplier-specific Price Meters**) At any time $t \geq 0$, Supplier i observes a continuously decreasing price meter $P_i(t)$, defined as follows:

$$P_i(t) = \psi_i^{-1}\left[\min\left\{\psi_i(\bar{c}_i), \max_j \psi_j(\bar{c}_j) - rt\right\}\right], \tag{2.4}$$

  where r is the rate at which the price is reduced and is a parameter chosen by the buyer.

- **Step 2**: (**Initialization**) Setting $\hat{\mathcal{N}} = \mathcal{N}$, $\hat{f} = f$, $\hat{P}_i = P_i(0)$ and $Q_i = 0$ for all $i \in \hat{\mathcal{N}}$, the buyer implements Step 3.

- **Step 3**: (**Possible Awards**) The buyer awards $\delta_i = \hat{f}(\hat{\mathcal{N}}) - \hat{f}(\hat{\mathcal{N}} \setminus \{i\})$ units to each Supplier $i \in \hat{\mathcal{N}}$ at unit price equal to $\hat{P}_i$. After these awards are made, the buyer updates $Q_i = Q_i + \delta_i$ for all $i \in \hat{\mathcal{N}}$ and $\hat{f}(\mathcal{S}) = f(\mathcal{S}) - \sum_{j \in \mathcal{S}} Q_j$ for all $\mathcal{S} \subseteq \hat{\mathcal{N}}$. If $\hat{f}(\hat{\mathcal{N}}) = 0$, then the auction ends. Otherwise, the buyer implements Step 4.

- **Step 4**: (**Elimination**) The buyer reduces the price meters until a supplier, denoted by $\hat{i}$, leaves the auction. At this time instant, the buyer sets $\hat{\mathcal{N}} = \hat{\mathcal{N}} \setminus \{\hat{i}\}$ and $\hat{P}_i$ equal to the current price meter of Supplier i for all $i \in \hat{\mathcal{N}}$. The buyer returns to Step 3.

Throughout this mechanism (and also for the mechanisms in Sections 2.1.2 and 2.2.2), the incremental and cumulative awards given to the suppliers can also be displayed on their respective "quantity meters".

Remark 1: The auction is defined for those realizations of $\mathbf{c} = (c_1, c_2, \ldots, c_N)$ in which the virtual costs $\psi_i(c_i), i = 1, 2, \ldots, N$, are distinct (i.e., no ties). This happens with probability 1 because of our assumption that $\psi_i(\cdot)$ is a strictly increasing function and $F_i(\cdot)$ is continuous, for all i. However, the auction can also be modified to accommodate ties. Consider a time instant when multiple suppliers, say those belonging to a set $\hat{\mathcal{N}}_e$, leave simultaneously. Consider an arbitrary sequence of the suppliers in the set $\hat{\mathcal{N}}_e$. Starting with the first supplier and considering one supplier at a time in this sequence, that supplier is eliminated from the set $\hat{\mathcal{N}}$. After each elimination, incremental awards are given to the remaining suppliers. The price meters of the suppliers are temporarily frozen to their current values until all the suppliers from the set $\hat{\mathcal{N}}_e$ have been eliminated. The precise description of this tie-breaking process is as follows: Let $N_e = |\hat{\mathcal{N}}_e|$ and let $(\pi_1, \pi_2, \ldots, \pi_{N_e})$ be an arbitrary sequence of suppliers in the set $\hat{\mathcal{N}}_e$. Let the index k be initialized to 1. The buyer sets $\hat{\mathcal{N}} = \hat{\mathcal{N}} \setminus \{\pi_k\}$, and makes an incremental award of $\delta_i = \hat{f}(\hat{\mathcal{N}}) - \hat{f}(\hat{\mathcal{N}} \setminus \{i\})$ units to each Supplier $i \in \hat{\mathcal{N}}$ at a unit price equal to $\hat{P}_i$. After these awards are made, the buyer updates $Q_i = Q_i + \delta_i$ for all $i \in \hat{\mathcal{N}}$ and $\hat{f}(\mathcal{S}) = f(\mathcal{S}) - \sum_{j \in \mathcal{S}} Q_j$ for all $\mathcal{S} \subseteq \hat{\mathcal{N}}$. If $\hat{f}(\hat{\mathcal{N}}) = 0$, then the auction ends. Otherwise, the buyer sets $k = k + 1$, updates $\hat{\mathcal{N}}$, and repeats the award step above. This process continues for N_e iterations (i.e., until all the suppliers in the set $\hat{\mathcal{N}}_e$ have been eliminated). At this point, the buyer implements Step 4 and the auction continues.

Remark 2: We briefly discuss the auction proposed in Chen and Ishida (2013), since that auction also uses bidder-specific price meters. They show the optimality of their auction for the special problem of maximizing the expected revenue for a seller from selling *one unit* to multiple buyers. The translation of their mechanism to the reverse setting (a buyer procuring from multiple sellers) we study would use *increasing* price meters. Notice that the descending mechanisms proposed in Sections 2.1.2, 2.2.2, and 2.3.1 use *decreasing* price meters. This is not merely a cosmetic difference. Their auction (in a symmetric setting) is related to the first-price sealed-bid auction, where the bidders need to be quite sophisticated to perform the non-trivial computation necessary to derive equilibrium strategies. In contrast, our descending mechanisms are related to second-price sealed-bid auction, where it is a dominant strategy for each bidder to reveal his true cost.

Connection of Our Descending Mechanism to that in Goel et al. (2012): Goel et al. (2012) consider a forward setting in which the auctioneer sells multiple (homogenous) units of a good. Each bidder has a privately-known unit valuation for the good and a publicly-known budget that allows him to buy multiple units. The allocations given to the bidders are subject to feasibility constraints that can be expressed as a polymatroid. For this setting, the authors propose an ascending mechanism that achieves a Pareto-optimal solution, i.e., there is no other solution in which a player (the seller or a bidder) obtains a strictly higher utility and the remaining players are at least as better off. For the special case where the auctioneer sells a fixed quantity of the good and each bidder has a budget that is large enough to buy all units, their mechanism results in an *efficient allocation*. This mechanism, modified for the reverse setting, is the same as the descending mechanism proposed here, except for the following change in Step 1: The mechanism in Goel et al. (2012) has a common price meter for all the suppliers whereas our descending mechanism above has individualized price meters, one for each supplier.

The above connection plays a crucial role in proving an intermediate result in the following theorem.

Theorem 3. *The mechanism DM($\mathcal{N}$, f) is an optimal solution to M-PM(f).*

Proof of Theorem 3: Let $\mathbf{c} = (c_1, c_2, \ldots, c_N)$ denote the vector of unit costs of the suppliers, $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_N)$ denote the vector of allocations, and $\mathbf{M} = (M_1, M_2, \ldots, M_N)$ denote the vector of payments given to the suppliers. From the *Revelation Principle* (Myerson 1981), it is sufficient to restrict attention to the class of *direct revelation mechanisms*, i.e., mechanisms in which suppliers reveal their costs truthfully. Such a mechanism is defined by a collection of functions $\{\mathbf{Q}(\cdot), \mathbf{M}(\cdot)\}$, where $Q_i(\mathbf{c})$ is the allocation to Supplier i and $M_i(\mathbf{c})$ is the payment to Supplier i. Using standard arguments in mechanism design (see, e.g., Chapter 5 of Krishna 2002), an optimal solution $(\mathbf{Q}^*, \mathbf{M}^*)$ to M-PM(f) can be obtained as follows:

- For any $\mathbf{c}$, the allocation vector $\mathbf{Q}^*(\mathbf{c})$ solves:

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \sum_{i=1}^{N} \psi_i(c_i) Q_i \\ \text{s.t.} \quad & \mathbf{Q} \in P_f, \; \sum_{i=1}^{N} Q_i = f(\mathcal{N}). \end{aligned} \tag{MR-PM($\mathbf{c}$)}$$

- For every $\mathbf{c}$, the payment to Supplier i is:

$$M_i^*(\mathbf{c}) = c_i Q_i^*(\mathbf{c}) + \int_{c_i}^{\bar{c}_i} Q_i^*(t_i, \mathbf{c}_{-i}) \, dt_i$$

Then, it is easy to see that the following two claims complete the proof of Theorem 3:

Claim 1. *For every* $\mathbf{c}$ *and for every Supplier* i*, the quantity awarded to that supplier in DM(*$\mathcal{N}$*,* f*) is* $Q_i^*(\mathbf{c})$*.*

Claim 2. *The mechanisms DM(*$\mathcal{N}$*,*f*) and* $(\mathbf{Q}^*, \mathbf{M}^*)$ *result in the same expected payments to all suppliers. Therefore, they result in the same total expected procurement cost for the buyer.*

The proof of Claim 1 exploits the connection of our mechanism, DM($\mathcal{N}$, f) to that in Goel et al. (2012). The proof of Claim 2 invokes the Revenue Equivalence Theorem; see, e.g., Krishna (2002). The proofs of these two claims are technically involved, and are therefore provided in Appendix A. ■

Having established the optimality of the mechanism DM($\mathcal{N}$, f) for the mechanism design problem M-PM(f), we now use this result to show the validity of Theorems 1 and 2. To this end, we will construct a non-decreasing submodular set function f for each of our procurement problems M-CGP and M-BR and show that (a) the mechanism design problems M-PM(f) corresponding to these functions are equivalent to M-CGP and M-BR, respectively, and (b) the descending mechanism DM($\mathcal{N}$, f) corresponding to these functions specializes to the mechanisms in Section 2.1.2 and Section 2.2.2. This, along with Theorem 3 above, establishes Theorems 1 and 2.

2.3.2 Proof of Theorem 1

For any subset $\mathcal{S} \subseteq \mathcal{N}$, define $\mathcal{S}_k = \mathcal{S} \cap G_k$ for all $k \in \mathcal{M}$ and $\mathcal{S}^0 = \mathcal{S} \cap G^0$. Define the set function f_c as follows:

$$f_c(\mathcal{S}) = \min \left\{ Q, \ \sum_{k=1}^{M} \min \left(\sum_{j \in \mathcal{S}_k} \Gamma_j, \eta_k \right) + \sum_{j \in \mathcal{S}^0} \Gamma_j \right\}, \ \forall \mathcal{S} \subseteq \mathcal{N}. \tag{2.5}$$

Note that the fact that the groups are disjoint is exploited in the construction of f_c.

Theorem 1 follows as a consequence of the following claim:

Claim 3. *The mechanism DM($\mathcal{N}$, f_c) is the same as that in Section 2.1.2, and is an optimal solution to M-CGP.*

The proof of the above claim is provided in Appendix A. In proving this claim, we first establish the following two results: (a) The function f_c is non-decreasing and submodular. (b) The feasible region of M-CGP is identical to that of M-PM(f_c). These two results, Theorem 3, and the fact that the two problems, M-CGP and M-PM(f_c) have identical objective functions (namely, minimizing the total expected payment given to the suppliers), together imply that DM($\mathcal{N}$, f_c) results in an optimal solution to M-CGP. Using these results, we also establish that the incremental awards made in DM($\mathcal{N}$, f_c) are identical to those made in the descending mechanism in Section 2.1.2.

2.3.3 Proof of Theorem 2

Define the set function f_b as follows:

$$f_b(\mathcal{S}) = Q \min\{b|\mathcal{S}|, 1 - a(L - |\mathcal{S}|)^+\}, \ \forall \mathcal{S} \subseteq \mathcal{N}. \tag{2.6}$$

The fact that the constraints are symmetric across suppliers (i.e., the bounds on the allocations given to the selected suppliers are independent of their identities) is exploited in the construction of f_b.

Theorem 2 is an immediate consequence of the following claim:

Claim 4. *The mechanism DM($\mathcal{N}$, f_b) is the same as that in Section 2.2.2, and is an optimal solution to M-BR.*

The proof of the above claim is provided in Appendix A. Again, the first part of the claim requires proving that the incremental awards made in the two mechanisms are identical. The second part of the claim involves showing the following results: (a) The function f_b is non-decreasing and submodular. (b) Any feasible solution to M-BR is also feasible to M-PM(f_b). (c) The mechanism DM($\mathcal{N}$, f_b) results in a feasible solution to M-BR. These three results, Theorem 3, and the fact that the two problems, M-BR and M-PM(f_b) have identical objective functions (namely, minimizing the total expected payment given to the suppliers), together imply that DM($\mathcal{N}$, f_b) results in an optimal solution to M-BR.

2.3.4 Extension to Endogenous Procurement Quantity

Problem M-PM(f) can be extended to the case where the total quantity to be procured by the buyer is endogenous, and the buyer obtains a revenue of $R \cdot Q$ from consuming Q units. To solve this problem, we modify the descending mechanism DM($\mathcal{N}$, f) in Section 2.3.1 as follows:

- In Step 1, the definition of the price meters $P_i(t)$, for all $i \in \mathcal{N}$ is modified as follows:

$$P_i(t) = \psi_i^{-1}\left[\min\left\{R, \psi_i(\bar{c}_i), \max_j \psi_j(\bar{c}_j) - rt\right\}\right].$$

- In Step 2, we initialize $\hat{\mathcal{N}}$ as the set suppliers who are still in the auction after observing their initial price meters. If the starting price meters of some suppliers fall below their private costs, then these suppliers will not participate in the auction, i.e., leave immediately after they see their price meters. Consequently, the set $\hat{\mathcal{N}}$ may be different from the set $\mathcal{N}$.

We have verified that the descending mechanism DM($\mathcal{N}$, f) with the above modifications solves the endogenous-quantity extension of problem M-PM(f).

2.4 Extension to Concave Production Costs

In our analysis thus far, we have assumed that suppliers have linear production costs. In the procurement literature, researchers have investigated optimal mechanisms under non-linear production costs; see, e.g., Tunca et al. (2009) and Duenyas et al. (2013). A natural question arises: To what extent do our results, in particular the optimality of the descending mechanism in Section 2.3.1, hold when the suppliers incur non-linear production costs? In this section, we address this question by adopting the concave-cost structure in Duenyas et al. (2013) and show that our results in Section 2.3 hold with some minor modifications that are necessary to incorporate this concave-cost structure. We do require the additional restriction that the polymatroid P_f defining the feasibility constraints is *symmetric* (Groenevelt 1991), i.e.,

$$f(\mathcal{S}) = w(|\mathcal{S}|), \ \forall \mathcal{S} \subseteq \mathcal{N},$$

where $w(\cdot)$ is a non-decreasing and concave function with $w(0) = 0$. Thus, for any $\mathcal{S} \subseteq \mathcal{N}$, $f(\mathcal{S})$ depends only on the cardinality $|\mathcal{S}|$ and not on the identity of the individual suppliers constituting $\mathcal{S}$. Note that the polymatroid associated with problem M-BR (i.e., with $f = f_b$, defined as in (2.6) above) is a symmetric polymatroid, while the polymatroid associated with the problem M-CGP (i.e., with $f = f_c$, defined as in (2.5)) is not symmetric.

Consider the mechanism design problem M-PM(f) in Section 2.3 with the following modifications: (1) Each Supplier i incurs a *concave* production cost $c_i H(q)$ for producing q units, where c_i is that supplier's private information, and the function $H(\cdot)$ is an increasing and concave function common to all suppliers, satisfying $H(0) = 0$ (Duenyas et al. 2013). (2) The feasibility constraints imposed on the allocations given to the suppliers form a symmetric polymatroid P_f. We refer to this problem as M-PMC.

Consider the descending mechanism DM($\mathcal{N}$, f) in Section 2.3.1 with the following modification in the payments given to the suppliers in Step 3: The buyer awards $\delta_i = \hat{f}(\hat{\mathcal{N}}) - \hat{f}(\hat{\mathcal{N}} \setminus \{i\})$ units to each Supplier $i \in \hat{\mathcal{N}}$ with a corresponding payment of $\hat{P}_i[H(Q_i + \delta_i) - H(Q_i)]$. We refer to this mechanism as DMC($\mathcal{N}$, f).

Theorem 4. *The mechanism DMC($\mathcal{N}$, f) is an optimal solution to M-PMC.*

Proof of Theorem 4: Using standard arguments in mechanism design, an optimal solution $(\mathbf{Q}^*, \mathbf{M}^*)$ to M-PMC can be obtained as follows:

- For any $\mathbf{c}$, the allocation vector $\mathbf{Q}^*(\mathbf{c})$ solves:

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \sum_{i=1}^{N} \psi_i(c_i) H(Q_i) \\ \text{s.t.} \quad & \mathbf{Q} \in P_f, \ \sum_{i=1}^{N} Q_i = f(\mathcal{N}). \end{aligned} \tag{MR-PMC($\mathbf{c}$)}$$

- For every $\mathbf{c}$, the payment to Supplier i is:

$$M_i^*(\mathbf{c}) = c_i Q_i^*(\mathbf{c}) + \int_{c_i}^{\bar{c}_i} Q_i^*(t_i, \mathbf{c}_{-i}) \, dt_i$$

Then, the following two claims complete the proof:

Claim 5. *For every* $\mathbf{c}$ *and for every Supplier* i*, the quantity awarded to that supplier in* $DMC(\mathcal{N}, f)$ *is* $Q_i^*(\mathbf{c})$.

Claim 6. *The mechanisms* $DMC(\mathcal{N}, f)$ *and* $(\mathbf{Q}^*, \mathbf{M}^*)$ *result in the same expected payments to all the suppliers. Therefore, they result in the same total expected procurement cost for the buyer.*

The proof of Claim 5 is provided in Appendix A. We avoid providing the proof of Claim 6 since it is similar to that of Claim 2, with trivial modifications necessary to incorporate concave cost structure. ■

Remark 3: When suppliers have convex production costs, an optimal allocation rule can be obtained as a solution to an optimization problem that minimizes a convex objective function subject to a polymatroid feasible region. While this optimization problem has an interior optimal solution, the descending mechanism in Section 2.3.1 results in an allocation that is necessarily a vertex of the polymatroid, and therefore may be sub-optimal. Similarly, for the mechanism design problem in which the buyer has an increasing and concave revenue function and suppliers have linear production costs, the descending mechanism may be sub-optimal.

2.5 Conclusion and Future Research Directions

In real-world procurement problems, it is typical for buyers to impose bounds on the total amount to be sourced from individual suppliers and/or from subsets of suppliers. This chapter studies two such procurement problems, and derives descending auction mechanisms that are optimal. Moreover, we show that these mechanisms are special cases of a descending mechanism that is optimal for a more general mechanism design problem with constraints specified by a polymatroid. While in this general problem the buyer procures a fixed (exogenously-specified) quantity and suppliers have linear production costs, we also consider two extensions – one in which the procured amount is endogenous and the other in which suppliers have concave production costs – and again develop descending auction mechanisms that are optimal.

We briefly discuss possible research directions. For the procurement problems considered in Sections 2.1–2.4, we assumed that the buyer procures a fixed quantity of a product and the underlying distributions on the costs of the suppliers are independent and common-knowledge. These assumptions are critical for the optimality of the descending mechanisms proposed in this chapter. It will be interesting to extend this work to incorporate some of the following realistic scenarios: (1) The costs of the suppliers are correlated (which is often the case when suppliers source components from a common upstream-supplier), (2) The buyer needs to procure multiple units of many products, (3) The feasibility constraints on the allocations given to the suppliers do not form a polymatroid.

CHAPTER 3

ON A MODIFICATION OF THE VCG MECHANISM AND ITS OPTIMALITY[1]

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This chapter is organized as follows: In Sections 3.1 and 3.2, we formally define the optimal mechanism and the VCG mechanism for the procurement setting introduced in Chapter 1. In Sections 3.3 and 3.4, we define the VVCG mechanism and present conditions under which it is optimal. Section 3.5 presents an example that shows that VVCG can be suboptimal. We conclude our chapter in Section 3.6.

3.1 The Optimal Mechanism

For every $i \in \mathcal{N}$, let $\psi_i(c) = c + F_i(c)/f_i(c)$ be a "virtual" cost function for all $c \in [\underline{c}_i, \bar{c}_i]$. We make the standard assumption that $\psi_i(\cdot)$ is strictly increasing for all $i \in \mathcal{N}$. Let $\psi_i^{-1}(v)$ be the inverse of the function $\psi_i(\cdot)$ for all $i \in \mathcal{N}$; i.e., $\psi_i^{-1}(v)$ is equal to (i) $\underline{c}_i$ if $v \leq \psi_i(\underline{c}_i)$, (ii) c if $\exists c \in [\underline{c}_i, \bar{c}_i]$ such that $v = \psi_i(c)$, and (iii) $\bar{c}_i$ if $v \geq \psi_i(\bar{c}_i)$. Let $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ denote the vector of bids of the suppliers, $\mathbf{b}_{-i}$ denote the vector of bids excluding the bid of supplier $i \in \mathcal{N}$, $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_{N+1})$ denote the vector of allocations given to the suppliers, and $\mathbf{M} = (M_1, M_2, \ldots, M_{N+1})$ denote the vector of payments given to the suppliers. For a bid vector $\mathbf{b}$, let $\boldsymbol{\psi}(\mathbf{b}) = (\psi_1(b_1),\ \psi_2(b_2),\ \ldots,\ \psi_N(b_N))$ denote the vector of virtual bids of the suppliers.

Using standard arguments in mechanism design (see e.g., Myerson 1981), the following sealed-bid mechanism, denoted by $(\mathbf{Q}^{OPT}, \mathbf{M}^{OPT})$, is optimal under the above setting: For a given $\mathbf{b}$:

- The allocation vector $\mathbf{Q}^{OPT}(\mathbf{b}, R)$ solves:

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \sum_{i=1}^{N} \psi_i(b_i) Q_i + R \cdot Q_{N+1} \\ \text{s.t.} \quad & \mathbf{Q} \in \text{FEAS}. \end{aligned}$$

- The payment given to supplier $i \in \mathcal{N}$ is

$$M_i^{OPT}(\mathbf{b}, R) = b_i Q_i^{OPT}(\mathbf{b}, R) + \int_{b_i}^{\bar{c}_i} Q_i^{OPT}(z, \mathbf{b}_{-i}, R)\, dz \tag{M-OPT}$$

  and the payment given to supplier $N+1$ is $M_{N+1}^{OPT}(\mathbf{b}, R) = R \cdot Q_{N+1}^{OPT}(\mathbf{b}, R)$.

We refer to the above mechanism as OPT.

3.2 The VCG Mechanism

We now define the VCG mechanism (see e.g., Krishna 2002), denoted by $\{\mathbf{Q}^{VCG}, \mathbf{M}^{VCG}\}$. For a given $\mathbf{b}$:

- The allocation vector $\mathbf{Q}^{VCG}(\mathbf{b}, R)$ solves:

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \sum_{i=1}^{N} b_i Q_i + R \cdot Q_{N+1} \\ \text{s.t.} \quad & \mathbf{Q} \in \text{FEAS}. \end{aligned} \tag{Q-VCG}$$

- Define $H(\mathbf{b}, R) = \sum_{i=1}^{N} b_i Q_i^{VCG}(\mathbf{b}, R) + R \cdot Q_{N+1}^{VCG}(\mathbf{b}, R)$ as the optimal value of the objective function in (Q-VCG). Let $H_{-i}(\mathbf{b}, R) = H(\mathbf{b}, R) - b_i Q_i^{VCG}(\mathbf{b}, R)$ for all $i \in \mathcal{N}$. The payment given to supplier $i \in \mathcal{N}$ is

$$M_i^{VCG}(\mathbf{b}, R) = H(\bar{c}_i, \mathbf{b}_{-i}, R) - H_{-i}(\mathbf{b}, R)$$

  and the payment given to supplier $N+1$ is $M_{N+1}^{VCG}(\mathbf{b}, R) = R \cdot Q_{N+1}^{VCG}(\mathbf{b}, R)$.

Let us now consider the special case of procuring one unit in the absence of the outside option (i.e., $R = \infty$). Let $(\tilde{1})$ and $(\tilde{2})$ denote the supplier with the lowest and second-lowest bid, respectively. Also, let (1) and (2) denote the supplier with the lowest and second-lowest virtual bid, respectively. Then, in this special case, VCG procures the unit from supplier $(\tilde{1})$ and pays him $b_{(\tilde{2})}$; i.e., the highest amount supplier $(\tilde{1})$ can bid and still be the *lowest-bid supplier*. The optimal mechanism OPT procures the unit from supplier (1) and pays him the highest amount supplier (1) can bid and still be the *lowest-virtual-bid supplier*. That is, OPT pays supplier (1) the amount $\psi_{(1)}^{-1}[\psi_{(2)}(b_{(2)})]$; this is easy to derive using (M-OPT). Motivated by this parallel between VCG and OPT in this special case, we define a transformation of VCG which we refer to as the "virtual" VCG mechanism (VVCG).

3.3 The VVCG Mechanism

For a bid vector $\mathbf{b}$ and $i \in \mathcal{N}$, let $\boldsymbol{\psi}(\mathbf{b}) = \big(\psi_1(b_1), \psi_2(b_2), \ldots, \psi_N(b_N)\big)$ denote the vector of virtual bids of the suppliers and $\hat{\boldsymbol{\psi}}_i(\mathbf{b}) = \big(\psi_i^{-1}[\psi_1(b_1)],\ \psi_i^{-1}[\psi_2(b_2)],\ \ldots\ ,\ \psi_i^{-1}[\psi_N(b_N)]\big)$. The VVCG mechanism, denoted by $\{\mathbf{Q}^{VVCG}, \mathbf{M}^{VVCG}\}$, is a sealed-bid mechanism defined as follows:

- The allocation given to supplier $i \in \{1, 2, \ldots, N+1\}$ is

$$Q_i^{VVCG}(\mathbf{b}, R) = Q_i^{VCG}(\boldsymbol{\psi}(\mathbf{b}), R).$$

- The payment given to supplier $i \in \mathcal{N}$ is

$$M_i^{VVCG}(\mathbf{b}, R) = M_i^{VCG}(\hat{\boldsymbol{\psi}}_i(\mathbf{b}), \psi_i^{-1}(R)) \tag{M-VVCG}$$

  and the payment given to supplier $N+1$ is $M_{N+1}^{VVCG}(\mathbf{b}, R) = R \cdot Q_{N+1}^{VVCG}(\mathbf{b}, R)$.

For the special case of procuring one unit in the absence of the outside option, the VVCG mechanism simply reduces to the mechanism in which the supplier with the lowest virtual bid is selected and paid the highest amount that the supplier could bid to be selected. Thus, it is identical to OPT for this case.

3.4 Optimality of VVCG

Let g: $2^{\mathcal{N} \cup \{N+1\}} \to \mathbb{R}_+$ be a non-decreasing and submodular set function with $g(\emptyset) = 0$. That is, we have: (1) $g(\mathcal{S}) \leq g(\mathcal{T})$, for all $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{N} \cup \{N+1\}$ (non-decreasing), and (2) $g(\mathcal{S} \cup \{s\}) - g(\mathcal{S}) \geq g(\mathcal{T} \cup \{s\}) - g(\mathcal{T})$, for all $\mathcal{S} \subseteq \mathcal{T}$, $s \in \mathcal{N} \cup \{N+1\} \setminus \mathcal{T}$ (submodular). The set of polymatroid feasibility constraints are defined as follows:

$$P_g = \left\{ \mathbf{Q} \in R_+^{N+1} \;:\; \sum_{i \in \mathcal{S}} Q_i \leq g(\mathcal{S}) \; \forall \, \mathcal{S} \subseteq \mathcal{N} \cup \{N+1\}, \; \sum_{i=1}^{N+1} Q_i = g(\mathcal{N} \cup \{N+1\}) \right\}.$$

Examples of papers that consider polymatroid feasibility constraints in an auction framework include Bikhchandani et al. (2011), Goel et al. (2012) and Gupta et al. (2015). We now state our main result.

Theorem 5. *Let g be a non-decreasing and submodular set function with $g(\emptyset) = 0$. If $FEAS = P_g$, then the VVCG mechanism is identical to OPT, and therefore optimal.*

Proof of Theorem 5: Fix an arbitrary bid vector $\mathbf{b}$. By definition, $\mathbf{Q}^{VVCG}(\mathbf{b}, R) = \mathbf{Q}^{OPT}(\mathbf{b}, R)$ and $M_{N+1}^{VVCG}(\mathbf{b}, R) = M_{N+1}^{OPT}(\mathbf{b}, R)$. Therefore, Theorem 5 can be established if we show that $M_i^{VVCG}(\mathbf{b}, R) = M_i^{OPT}(\mathbf{b}, R)$ for all $i \in \mathcal{N}$ under polymatroidal feasibility constraints. To prove this, we exploit the fact that $\mathbf{Q}^{VCG}(\boldsymbol{\psi}(\mathbf{b}), R)$ for polymatroidal feasibility constraints can be explicitly obtained by the well-known greedy algorithm of Edmonds (1971) and has the following property: *The solution depends only the ranks of the suppliers in the ordered vector* $(\boldsymbol{\psi}(\mathbf{b}), R)$.

For ease of notational exposition, let us assume a proxy bid $b_{N+1} = R$ and correspondingly, a virtual bid $\psi_{N+1}(b_{N+1}) = R$ for the outside option supplier. Let (i) denote the index of the supplier with rank i in the ordered vector $(\boldsymbol{\psi}(\mathbf{b}), R)$, i.e., $\psi_{(1)}(b_{(1)}) \leq \psi_{(2)}(b_{(2)}) \leq \ldots \leq \psi_{(N+1)}(b_{(N+1)})$. We are now ready to obtain the allocation vector $\mathbf{Q}^{VCG}(\boldsymbol{\psi}(\mathbf{b}), R)$ using the greedy algorithm: Let $A_0 = \emptyset$ and $A_i = \{(1), (2), \ldots, (i)\}$ for every i. Then $Q^{VCG}_{(i)}(\boldsymbol{\psi}(\mathbf{b}), R) = g(A_i) - g(A_{i-1})$ for all i.

Consider an arbitrary supplier in the set $\mathcal{N}$ with rank i in the ordered vector $(\boldsymbol{\psi}(\mathbf{b}), R)$. Let k be the rank of this supplier in the ordered vector $(\boldsymbol{\psi}(\bar{c}_{(i)}, \mathbf{b}_{-(i)}), R)$. Then, we have the following cases:

- If $\psi_{(i)}(\bar{c}_{(i)}) < \psi_{(i+1)}(b_{(i+1)})$, then $k = i$.
- If $\exists j \in \{i+1, i+2, \ldots, N\}$ such that $\psi_{(j)}(b_{(j)}) \leq \psi_{(i)}(\bar{c}_{(i)}) < \psi_{(j+1)}(b_{(j+1)})$, then $k = j$.
- If $\psi_{(i)}(\bar{c}_{(i)}) \geq \psi_{(N+1)}(b_{(N+1)})$, then $k = N + 1$.

Define $B_j = A_j \backslash \{(i)\}$ for any $j \geq i$. We have the following:

$$
\begin{aligned}
& M^{VVCG}_{(i)}(\mathbf{b}, R) - b_{(i)} Q^{VVCG}_{(i)}(\mathbf{b}, R) \\
&= M^{VCG}_{(i)}(\hat{\boldsymbol{\psi}}_{(i)}(\mathbf{b}), \psi^{-1}_{(i)}(R)) - b_{(i)} Q^{VCG}_{(i)}(\boldsymbol{\psi}(\mathbf{b}), R) \\
&= H(\hat{\boldsymbol{\psi}}_{(i)}(\bar{c}_{(i)}, \mathbf{b}_{-(i)}), \psi^{-1}_{(i)}(R)) - H_{-(i)}(\hat{\boldsymbol{\psi}}_{(i)}(\mathbf{b}), \psi^{-1}_{(i)}(R)) - b_{(i)} Q^{VCG}_{(i)}(\hat{\boldsymbol{\psi}}_{(i)}(\mathbf{b}), \psi^{-1}_{(i)}(R)) \quad (3.1) \\
&= \bar{c}_{(i)} Q^{VCG}_{(i)}(\hat{\boldsymbol{\psi}}_{(i)}(\bar{c}_{(i)}, \mathbf{b}_{-(i)}), \psi^{-1}_{(i)}(R)) - b_{(i)} Q^{VCG}_{(i)}(\hat{\boldsymbol{\psi}}_{(i)}(\mathbf{b}), \psi^{-1}_{(i)}(R)) \; + \\
&\qquad \sum_{j \neq (i)} \psi^{-1}_{(i)}[\psi_j(b_j)] \left[Q^{VCG}_{j}(\hat{\boldsymbol{\psi}}_{(i)}(\bar{c}_{(i)}, \mathbf{b}_{-(i)}), \psi^{-1}_{(i)}(R)) - Q^{VCG}_{j}(\hat{\boldsymbol{\psi}}_{(i)}(\mathbf{b}), \psi^{-1}_{(i)}(R)) \right] \\
&= \bar{c}_{(i)} Q^{VCG}_{(i)}(\hat{\boldsymbol{\psi}}_{(i)}(\bar{c}_{(i)}, \mathbf{b}_{-(i)}), \psi^{-1}_{(i)}(R)) - b_{(i)} Q^{VCG}_{(i)}(\hat{\boldsymbol{\psi}}_{(i)}(\mathbf{b}), \psi^{-1}_{(i)}(R)) \; + \\
&\qquad \sum_{j=i+1}^{k} \psi^{-1}_{(i)}[\psi_{(j)}(b_{(j)})] \left[Q^{VCG}_{(j)}(\hat{\boldsymbol{\psi}}_{(i)}(\bar{c}_{(i)}, \mathbf{b}_{-(i)}), \psi^{-1}_{(i)}(R)) - Q^{VCG}_{(j)}(\hat{\boldsymbol{\psi}}_{(i)}(\mathbf{b}), \psi^{-1}_{(i)}(R)) \right] \quad (3.2) \\
&= \bar{c}_{(i)}[g(A_k) - g(B_k)] - b_{(i)}[g(A_i) - g(A_{i-1})] \; + \\
&\qquad \sum_{j=i+1}^{k} \psi^{-1}_{(i)}[\psi_{(j)}(b_{(j)})] \left[g(B_j) - g(B_{j-1}) - g(A_j) + g(A_{j-1}) \right] \\
&= \left(\psi^{-1}_{(i)}[\psi_{(i+1)}(b_{(i+1)})] - b_{(i)} \right) \left(g(A_i) - g(A_{i-1}) \right) \; +
\end{aligned}
$$

$$\left(\bar{c}_{(i)} - \psi_{(i)}^{-1}[\psi_{(k)}(b_{(k)})]\right)\left(g(A_k) - g(B_k)\right) \ + $$
$$\sum_{j=i+2}^{k} \left(\psi_{(i)}^{-1}[\psi_{(j)}(b_{(j)})] - \psi_{(i)}^{-1}[\psi_{(j-1)}(b_{(j-1)})]\right)\left(g(A_{j-1}) - g(B_{j-1})\right)$$

$$= \int_{b_{(i)}}^{\psi_{(i)}^{-1}[\psi_{(i+1)}(b_{(i+1)})]} Q_{(i)}^{VCG}\big(\boldsymbol{\psi}(z, \mathbf{b}_{-(i)}), R\big)\ dz \ + \ \int_{\psi_{(i)}^{-1}[\psi_{(k)}(b_{(k)})]}^{\bar{c}_{(i)}} Q_{(i)}^{VCG}\big(\boldsymbol{\psi}(z, \mathbf{b}_{-(i)}), R\big)\ dz \ + $$
$$\sum_{j=i+2}^{k} \left[\int_{\psi_{(i)}^{-1}[\psi_{(j-1)}(b_{(j-1)})]}^{\psi_{(i)}^{-1}[\psi_{(j)}(b_{(j)})]} Q_{(i)}^{VCG}\big(\boldsymbol{\psi}(z, \mathbf{b}_{-(i)}), R\big)\ dz\right]$$

$$= \int_{b_{(i)}}^{\bar{c}_{(i)}} Q_{(i)}^{VCG}\big(\boldsymbol{\psi}(z, \mathbf{b}_{-(i)}), R\big)\ dz$$

$$= \int_{b_{(i)}}^{\bar{c}_{(i)}} Q_{(i)}^{VVCG}(z, \mathbf{b}_{-(i)}, R)\ dz.$$

The equality (3.1) follows from the facts that the allocation $\mathbf{Q}^{VCG}(\cdot)$ is obtained from the greedy algorithm and the function $\psi_{(i)}^{-1}(\cdot)$ is increasing. Also, for every $j \in \{1, 2, \dots, i-1, k+1, k+2, \dots, N+1\}$, we have $Q_{(j)}^{VCG}(\boldsymbol{\psi}(\bar{c}_{(i)}, \mathbf{b}_{-(i)}), R) = Q_{(j)}^{VCG}(\boldsymbol{\psi}(\mathbf{b}), R) = g(A_j) - g(A_{j-1})$. This results in (3.2).

To summarize, we have $M_{(i)}^{VVCG}(\mathbf{b}, R) = b_{(i)} Q_{(i)}^{VVCG}(\mathbf{b}, R) + \int_{b_{(i)}}^{\bar{c}_{(i)}} Q_{(i)}^{VVCG}(z, \mathbf{b}_{-(i)}, R)\ dz$. This along with the fact that $\mathbf{Q}^{VVCG}(\mathbf{b}, R) = \mathbf{Q}^{OPT}(\mathbf{b}, R)$ imply that $M_{(i)}^{VVCG}(\mathbf{b}, R) = M_{(i)}^{OPT}(\mathbf{b}, R)$. This completes the proof of Theorem 5. ■

Corollary 1. *For the special case of symmetric suppliers (i.e., $F_i = F$, $\underline{c}_i = \underline{c}$ and $\bar{c}_i = \bar{c}$ for all i), polymatroidal feasibility constraints, and in the absence of the outside option, the VCG mechanism results in the same outcome as the VVCG mechanism, and hence optimal.*

An analogous result to Corollary 1 is mentioned in Dughmi et al. (2012) in which a seller sells multiple units of a good to N *symmetric* buyers with a restriction that the feasible set of winners form a matroid.

Corollary 1 identifies a set of conditions under which VVCG is identical to the VCG mechanism and is optimal. In the following result, we identify another set of sufficient conditions under which VVCG is identical to the VCG mechanism and is optimal.

Theorem 6. *If the costs of the suppliers in the set $\mathcal{N}$ are i.i.d. draws from a uniform distribution with support $[0, \bar{c}]$ and there is no outside option, then the VVCG mechanism is identical to the VCG mechanism and is optimal for an arbitrary set of feasibility constraints.*

Proof of Theorem 6: Under the conditions of Theorem 6, the allocations and payments given to the suppliers in the VVCG mechanism are specified as follows: For every supplier i and bid vector $\mathbf{b}$, $Q_i^{VVCG}(\mathbf{b}) = Q_i^{VCG}(2 \cdot \mathbf{b})$ and $M_i^{VVCG}(\mathbf{b}) = M_i^{VCG}(\mathbf{b})$. Since $\mathbf{Q}^{VCG}(2 \cdot \mathbf{b}) = \mathbf{Q}^{VCG}(\mathbf{b})$ for all $\mathbf{b}$, it is easy to see that the VVCG mechanism reduces to the VCG mechanism. To prove the optimality of the VCG mechanism, we show that the total expected payment given to the suppliers in this mechanism is equal to the total expected payment given in the optimal mechanism OPT.

Note that $\mathbf{Q}^{OPT}(\mathbf{b}) = \mathbf{Q}^{VCG}(2 \cdot \mathbf{b}) = \mathbf{Q}^{VCG}(\mathbf{b})$ for all $\mathbf{b}$. Furthermore, it is well-known that both VCG and OPT are truth-telling mechanisms. Consider an arbitrary supplier $i \in \mathcal{N}$. Let $m_i^{VCG}(c_i) = \mathbb{E}_{\mathbf{c}_{-i}}[M_i^{VCG}(\mathbf{c})]$ and $m_i^{OPT}(c_i) = \mathbb{E}_{\mathbf{c}_{-i}}[M_i^{OPT}(\mathbf{c})]$ for all $c_i \in [0, \bar{c}]$. Using Revenue Equivalence Theorem; see, e.g., Proposition 5.2 of Krishna (2002), we have $m_i^{VCG}(c_i) - m_i^{OPT}(c_i) = m_i^{VCG}(\bar{c}) - m_i^{OPT}(\bar{c})$ for all c_i. Note that $M_i^{VCG}(\bar{c}, \mathbf{c}_{-i}) = \bar{c} \cdot Q_i^{VCG}(\bar{c}, \mathbf{c}_{-i})$ and $M_i^{OPT}(\bar{c}, \mathbf{c}_{-i}) = \bar{c} \cdot Q_i^{OPT}(\bar{c}, \mathbf{c}_{-i})$. Since $\mathbf{Q}^{VCG}(\mathbf{c}) = \mathbf{Q}^{OPT}(\mathbf{c})$ for all $\mathbf{c}$, we have $M_i^{VCG}(\bar{c}, \mathbf{c}_{-i}) = M_i^{OPT}(\bar{c}, \mathbf{c}_{-i})$, and consequently, $m_i^{VCG}(\bar{c}) = m_i^{OPT}(\bar{c})$. Thus, $m_i^{VCG}(c_i) = m_i^{OPT}(c_i)$ for all c_i and $i \in \mathcal{N}$. Finally, $\mathbb{E}\left[\sum_i m_i^{VCG}(c_i)\right] = \mathbb{E}\left[\sum_i m_i^{OPT}(c_i)\right]$, which completes the proof of Theorem 6. ■

In the next section, we show that if the set FEAS is not a polymatroid, then the VVCG mechanism may not result in an optimal outcome.

3.5 Sub-Optimality of VVCG: An Example

Lemma 1. *Consider the following example of a buyer and two suppliers: Supplier 1 has a privately-known unit cost c_1, which is a realization of a uniform distribution with support $[0, \bar{c}]$. Supplier 2 has a publicly-known unit cost R. Let $m > 1$ and $R \geq 2m\bar{c}$. There are only two possible feasible allocations: $(0, 1)$ and $(m, 0)$. Then, for this example, the VVCG mechanism is not an optimal mechanism for the buyer.*

Proof of Lemma 1: To prove Lemma 1, we show that the total expected payment given to the suppliers in the VVCG mechanism is higher than the total expected payment given in the optimal mechanism. Here, the solution $\mathbf{Q}^{VCG}(b_1, R)$ defined in (Q-VCG) is obtained as follows:

$$\mathbf{Q}^{VCG}(b_1, R) = \begin{cases} (0, 1), & b_1 > R/m, \\ (m, 0), & b_1 \leq R/m. \end{cases}$$

To obtain the total expected payment in the VVCG mechanism, we first identify the optimal bidding strategy of supplier 1. Let $b_1^*(c_1) = \mathrm{argmax}_{b_1} U_1^{VVCG}(b_1, c_1, R)$ be the optimal bidding strategy of supplier 1, where $U_1^{VVCG}(b_1, c_1, R)$ is the utility of that supplier when he submits the bid b_1. Using the definition of the VVCG mechanism and the assumption that $R \geq 2m\bar{c}$, the utility of supplier 1 can be expressed as follows:

$$
\begin{aligned}
&U_1^{VVCG}(b_1, c_1, R) = M_1^{VVCG}(b_1, R) - c_1 Q_1^{VVCG}(b_1, R) \\
&= M_1^{VCG}(b_1, \psi_1^{-1}(R)) - c_1 Q_1^{VCG}(\psi_1(b_1), R) \\
&= \bar{c} Q_1^{VCG}(\bar{c}, \psi_1^{-1}(R)) + \psi_1^{-1}(R) Q_2^{VCG}(\bar{c}, \psi_1^{-1}(R)) - \\
&\quad\quad \psi_1^{-1}(R) Q_2^{VCG}(b_1, \psi_1^{-1}(R)) - c_1 Q_1^{VCG}(\psi_1(b_1), R) \\
&= \bar{c} Q_1^{VCG}(\bar{c}, \bar{c}) + \bar{c} Q_2^{VCG}(\bar{c}, \bar{c}) - \bar{c} Q_2^{VCG}(b_1, \bar{c}) - c_1 Q_1^{VCG}(2b_1, R) \\
&= \bar{c} - \bar{c} Q_2^{VCG}(b_1, \bar{c}) - c_1 Q_1^{VCG}(2b_1, R).
\end{aligned}
$$

Thus,

$$
U_1^{VVCG}(b_1, c_1, R) = \begin{cases} \bar{c} - mc_1, & b_1 < \bar{c}/m \\ -mc_1, & \bar{c}/m \leq b_1 \leq R/(2m) \\ 0, & b_1 > R/(2m). \end{cases}
$$

It is trivial to see that $b_1^*(c_1) < \bar{c}/m$ if $mc_1 \leq \bar{c}$, and $b_1^*(c_1) > R/(2m)$ otherwise. As a result, we have: (1) $M_1^{VVCG}(b_1^*(c_1), R) = \bar{c}$ if $c_1 \leq \bar{c}/m$, and 0 otherwise. (2) $M_2^{VVCG}(b_1^*(c_1), R) = 0$ if $c_1 \leq \bar{c}/m$, and R otherwise. Thus, the total expected payment given to the suppliers in VVCG is

$$
\begin{aligned}
\mathbb{E}_{c_1}[M_1^{VVCG}(b_1^*(c_1), R) + M_2^{VVCG}(b_1^*(c_1), R)] &= \bar{c}\ p(c_1 \leq \bar{c}/m) + R\ p(c_1 > \bar{c}/m) \\
&= \frac{\bar{c}}{m} + \left(1 - \frac{1}{m}\right) R. \quad (3.3)
\end{aligned}
$$

To compare this with the optimal total expected payment given to the suppliers, we use the OPT mechanism introduced in Section 3.1 to obtain an optimal solution for this example as follows: (1) $\mathbf{Q}^{OPT}(c_1, R) = (m, 0)$. (2) $\mathbf{M}^{OPT}(c_1, R) = (m\bar{c}, 0)$. Thus, the total expected payment given to the suppliers in OPT is $\mathbb{E}_{c_1}[M_1^{OPT}(c_1, R) + M_2^{OPT}(c_1, R)] = m\bar{c}$. This along with (3.3) gives:

$$
\begin{aligned}
&\mathbb{E}_{c_1}[M_1^{VVCG}(b_1^*(c_1), R) + M_2^{VVCG}(b_1^*(c_1), R)] - \mathbb{E}_{c_1}[M_1^{OPT}(c_1, R) + M_2^{OPT}(c_1, R)] \\
&= \frac{(m-1)[R - (m+1)\bar{c}]}{m},
\end{aligned}
$$

which is strictly positive under our assumptions that $R \geq 2m\bar{c}$ and $m > 1$. This proves Lemma 1. ■

3.6 Concluding Remarks

This chapter studies the VVCG mechanism, and investigates its optimality in the context of procurement. We show that VVCG is optimal if the feasibility constraints that govern the allocations given to the suppliers form a polymatroid. Using an example, we also demonstrate that if the feasibility constraints do not define a polymatroid, then VVCG may not be optimal.

CHAPTER 4

DISTRESSED SELLING BY FARMERS: MODEL, ANALYSIS, AND USE IN POLICY-MAKING

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This chapter is organized as follows: In Sections 4.1 and 4.2, we discuss the contributions of our work and the related literature. In Section 4.3, we present our finite-horizon stochastic DP on distressed selling and obtain an optimal policy for this DP. Section 4.4 analyzes the infinite-horizon approximation. For the special case of exponentially-distributed procurement capacities, we use the approximation to obtain closed-form expressions for the main quantities of interest, namely the volume of distressed sales and the loss in welfare of the farmers. Sections 4.5 and 4.6 test our analysis on real data and suggest ways in which it can be used to evaluate policy interventions. We conclude our chapter in Section 4.7 and discuss future research directions.

4.1 Our Contributions

The two main contributions of our work are as follows:

1. **Model, Validation on Real Data, and its Use in Policy-Making:** On the modeling front, our contribution is to build a tractable model that captures the salient features of the ground realities that cause distressed selling. Using real data on the production of rice and its procurement by the government of India under the support-price program, we establish the accuracy of our model's prediction on the volume of distressed sales. We then show how our model and its solution can serve as a simple and useful tool for policy-makers to capture the relative impact of the improvements in the main factors affecting distressed sales, namely the mean and variability of the procurement capacity, the holding cost incurred by the farmers, and the lack of affordable credit.

2. **Technical Analysis:** On the technical front, our contribution is twofold: (a) The real-world setting of distressed selling results in a finite-horizon stochastic dynamic program that is both interesting and non-trivial to solve. We obtain an optimal policy for this problem as well as for its infinite-horizon approximation, and establish an attractive performance guarantee on the closeness of the approximation. (b) We obtain closed-form expressions for the optimal policy – and for several quantities of interest to planners – for the special case of exponentially-distributed procurement capacities, which we show fit the real data well. The availability of these closed-form expressions increases the potential for the use of our work as a policy-making tool.

4.2 Literature Review

Broadly, our work belongs to the growing stream of research in Operations Management on agricultural operations. Below, we discuss a few papers in this domain.

An et al. (2015) consider a situation in which farmers choose to form an aggregation, which engages in Cournot competition with the rest of the individual farmers. The authors develop five different models to individually capture the impact of reduced cost, increased process yield, increased brand awareness, reduced intermediaries, and reduced price uncertainty, on the decisions of the farmers to join the aggregation and their payoffs. They show that a farmer joins an aggregation only when the size of the aggregation is below a certain threshold. Dawande et al. (2013) propose mechanisms for the socially-optimal distribution of surface water between primary farms (that are close to the water-sources) and secondary farms (that are relatively farther). Huh and Lall (2013) develop stochastic programming models to analyze a farmer's decision on the set of crops planted, the amount of irrigation water acquired, and the allocation of water to the selected crops, under uncertainty in market price and rainfall.

There are several papers that investigate the provision of valuable information to farmers and its impact on their equilibrium behavior and profits. For example, Chen et al. (2015) consider a problem in which there are multiple heterogeneous farmers, each endowed with a known production capability. Farmers can post questions on a forum to improve their production capabilities. A representative expert monitors these questions and answers them depending upon his availability. A core user (a knowledgeable farmer) can choose to be silent or responsive to such questions. If the core user chooses to respond, then he can strategically decide the extent or informativeness of his response. The authors show that, in equilibrium, the core user never provides an answer that is more informative than the answer provided by the expert. Chen and Tang (2015) construct a stylized model in which each farmer receives a noisy public signal and a noisy private signal to estimate the market price uncertainty. They analyze the farmers' production decisions in Cournot competition and show that the private signals always improve farmers' welfare. On the other hand, in the presence of private signals, there are instances when the public signal does not create value for the farmers. Parker et al. (2015) conduct an empirical study to show that information and communication technologies (such as mobile phone networks) reduce price dispersion of crops in rural markets. Tang et al. (2015) investigate the role of market information (e.g., market demand) and agricultural advice (e.g., on reducing production cost and/or improving process yield) when farmers engage in Cournot competition under uncertain market demand and uncertain process yield.

Chen et al. (2013) develop a single-period model in which ITC, an Indian conglomerate, can provide training to so-called "contracted" and "non-contracted" farmers to improve their productivity. The contracted farmers can supply their produce either directly to ITC or in the open market, whereas, the non-contracted farmers sell their produce only in the open market. The authors analyze farmers' strategic production and delivery decisions and show that the contracted farmers always sell directly to ITC. Furthermore, they find that, in many cases, it is in ITC's best interest to provide training to the non-contracted farmers as well.

There is a growing interest in analyzing governmental subsidy programs and their impact on farmers' welfare. For example, Alizamir et al. (2016) develop a model to analyze the effect of two specific U.S. government subsidy programs – Price Loss Coverage (PLC) and Agricultural Risk Coverage (ARC) – on farmers' planting decisions and social welfare. The PLC program insures farmers against losses due to low market prices whereas the ARC program insures farmers against losses due to low revenues. Most of their analysis considers a single farmer who has the option to enroll either in the PLC or ARC program and needs to decide the quantity of land for planting a crop at the beginning of the growing season. Yield uncertainty is realized at the end of the growing season, which together with the number of acres planted for the crop determines its market price. Our work also contributes to the literature on the operational analysis of socially-responsible schemes; see e.g., Cohen et al. (2015), Lim et al. (2015) and Muthulingam et al. (2013).

From a methodological standpoint, our work is also related to the wide stream of research on inventory management (see, e.g., Porteus 2002) and commodity operations (see, e.g., Secomandi 2010; Devalkar et al. 2011) that uses dynamic programming to obtain an optimal policy structure. In this body of work, typically a manufacturer is considered who makes ordering and selling decisions in order to maximize his expected payoff. In contrast, in the problem we study, starting with an initial inventory level, the farmers face random selling quotas and only make selling decisions in order to maximize their expected payoffs.

4.3 Model and Analysis

We begin by defining our model and formulating a stochastic DP in Section 4.3.1. Then, in Section 4.3.2, we obtain the structure of an optimal policy of this DP.

4.3.1 Problem Definition

The farming community[1] (hereafter, F) wants to sell a harvest of quantity Q in a selling season that consists of T periods. We use $t \in \{1, 2, \ldots, T\}$ as the index of a period. F can sell its produce either to the government (hereafter, G) at the support price S, or to an outside agent (hereafter, A) at a random unit price W_t. F incurs a per-period holding cost h for every unit of inventory on hand at the end of each period and a per-period discount factor $\alpha \in (0, 1)$. G is endowed with a random procurement capacity Y_t in period t. We assume that $\{(Y_t, W_t) : t = 1, 2, \ldots, T\}$ is a sequence of i.i.d. random vectors[2] with support $\mathcal{Y} \times \mathcal{W}$, where $\mathcal{Y} = [0, \overline{y}]$ and $\mathcal{W} = [\underline{w}, \overline{w}]$. We use (Y, W) to denote a generic pair (Y_t, W_t) and (y, w) to denote a realization of (Y, W). Let ϕ denote the marginal density function for Y. We assume that ϕ is strictly positive in the support $\mathcal{Y}$. The salvage value of any unsold quantity remaining with F at the end of the season is zero[3]. To avoid trivial outcomes, we make the realistic assumption that $\alpha S \geq \overline{w} + h$.

The sequence of events in any period $t \in \{1, 2, \ldots, T\}$ in F's problem is as follows:

1. F observes x_t, the inventory at the beginning of period t, y_t, the realization of Y_t (the procurement capacity of G in period t) and w_t, the realization of W_t (the unit price of A in period t). The state in period t is (x_t, y_t, w_t).

2. F makes selling decisions: (1) q_t^G is the amount sold to G in period t, and (2) q_t^A is the amount sold to A in period t. These quantities satisfy the following requirements: (a) $q_t^G, q_t^A \geq 0$, (b) $q_t^G \leq y_t$, (c) $q_t^G + q_t^A \leq x_t$. That is, the amounts sold to G and A in period t are non-negative, the amount sold to G in period t is at most its realized capacity in that period, and the total amount sold to G and A in period t is at most the inventory at the beginning of that period.

---

[1]To study the impact of operational factors that include limited and uncertain procurement capacity at GPCs, we consider a single-farmer model and abstract away from the effect of competition among farmers. Single-farmer models are common in the O.M. literature; see e.g., Alizamir et al. (2016) and Huh and Lall (2013).

[2]The i.i.d. assumption is purely used for analytical tractability. In particular, the optimal policy for our finite-horizon model holds even without this assumption. However, for our infinite-horizon approximation, we require this assumption to obtain an optimal policy structure.

[3]This assumption is only to ensure that F does not deliberately keep unsold quantity at the end of the selling season. For instance, the analysis remains unaffected as long as the salvage value is no more than $\underline{w}$.

Thus, we have the identity

$$x_t = Q - \sum_{s=1}^{t-1} (q_s^G + q_s^A).$$

A feasible policy is defined as a rule that uses only the information up until period t to specify the quantities q_t^G and q_t^A. Let Π denote the set of all feasible policies, and let π be an element of the set Π. The goal of F is to design a feasible policy that maximizes the expected present value of its profits over periods $1, 2, \ldots, T$; that is

$$\mathbb{E}\left[\sum_{t=1}^{T} \alpha^{t-1} \left(Sq_t^G + W_t q_t^A - hx_{t+1}\right)\right],$$

where the expectation is taken with respect to the random procurement capacities of G and the unit prices of A in periods $1, 2, \ldots, T$. We refer to this problem as P_T. Briefly, F faces the following trade-off: On the one hand, selling a unit to A guarantees an immediate cash inflow and ensures that no more costs are incurred in holding this unit in inventory. On the other hand, holding on to a unit gives the possibility of obtaining a higher revenue at a later (random) time. An optimal policy is a feasible policy that maximizes the expected present value of the profits of F over periods $1, 2, \ldots, T$, thereby balancing the above-mentioned trade-off. The amount of distressed sales is the total quantity sold to A over all the periods.

We now proceed to formulate Problem P_T as a stochastic dynamic program. Let $f_{t,T}(x, y, w)$ denote the optimal expected present value in period t of the profits received by F over periods $t, t+1, \ldots, T$, given that $x_t = x$, $Y_t = y$ and $W_t = w$. Bellman's equation for this problem is as follows: For every $t \in \{1, 2, \ldots, T\}$, $x \geq 0$ and $(y, w) \in \mathcal{Y} \times \mathcal{W}$, we have

$$\begin{aligned} f_{t,T}(x, y, w) = \max_{q^G, q^A} \quad & Sq^G + wq^A - h(x - q^G - q^A) \\ & + \alpha \mathbb{E}\left[f_{t+1,T}(x - q^G - q^A, Y_{t+1}, W_{t+1})\right] \qquad (4.1) \\ \text{s.t.} \quad & q^A \geq 0, \quad 0 \leq q^G \leq y, \quad q^A + q^G \leq x \ , \qquad (4.2) \end{aligned}$$

where $f_{T+1,T}(x, y, w) = 0$ for all $x \geq 0$, and $(y, w) \in \mathcal{Y} \times \mathcal{W}$.

Next, we derive some results on the optimal profit and the optimal policy for F. Throughout this chapter, we use increasing/decreasing in the weak sense.

4.3.2 Optimal Policy Structure

Lemma 2. *For every $t \in \{1, 2, \ldots, T\}$, and $(y, w) \in \mathcal{Y} \times \mathcal{W}$, the function $f_{t,T}(x, y, w)$ is concave and increasing in x; that is, as the available inventory, x, increases, the optimal profit increases whereas the marginal value of a unit decreases. Moreover, the marginal value of a unit is smaller than S; that is, $\frac{\partial}{\partial x} f_{t,T}(x, y, w) \leq S$ holds for all (t, x, y, w).*

The proof of the above result and all the subsequent results are provided in Appendix B. Our next result is the following. Since the agent's price is always lower than the government's support price, it seems intuitive that F should sell the maximum possible quantity to G in every period. We show that this is optimal.

Lemma 3. *The set* Π^G *contains an optimal policy for Problem* P_T*, where* $\Pi^G \subseteq \Pi$ *denotes the class of feasible policies in which F sells the maximum possible quantity to G in every period. That is, there exists an optimal policy under which* $q_t^G = \min\{x_t, Y_t\}$ *for all* t*.*

Aided by Lemma 3 which characterizes the optimal quantities sold to G, we will next characterize the optimal quantities sold to A. We need some preliminary analysis and definitions first.

Let $\mathbf{v}$ denote a set of functions $\{v_{t,T}(w) : t = 1, 2, \ldots, T, w \in \mathcal{W}\}$. The *sell-down-to* $\mathbf{v}$ policy, denoted by $\pi^S(\mathbf{v})$, is defined as follows: It sells the maximum possible quantity to G in every period. For every period t, if the inventory in period t after selling to G exceeds the *threshold* $v_{t,T}(W_t)$, it sells enough to A to bring the inventory down to the level $v_{t,T}(W_t)$. Mathematically, this policy is defined through the following relations:

$$q_t^G \;=\; \min(x_t, Y_t) \text{ and } q_t^A \;=\; (x_t - Y_t - v_{t,T}(W_t))^+ \quad \text{for all} \;\; t.$$

We will next specify a set of thresholds $\mathbf{v}^* = \{v_{t,T}^*(\cdot)\}$ and show that the sell-down-to $\mathbf{v}^*$ policy is optimal for P_T. To do this, we use Lemma 3 to rewrite Problem P_T as:

$$\begin{aligned} f_{t,T}(x,y,w) \;&=\; Sx \text{ if } x \le y\;, &(4.3)\\ &=\; Sy + \max_{\{v:0\le v\le x-y\}} w(x-y-v) - hv + \alpha\mathbb{E}[f_{t+1,T}(v, Y_{t+1}, W_{t+1})] \\ &\qquad \text{if } x > y. &(4.4)\end{aligned}$$

Here, the decision variable v denotes the amount of inventory that F carries over into the next period, i.e., period $t+1$. Define $v_{t,T}^*(w)$ as

$$v_{t,T}^*(w) = \operatorname{argmax}_{v \ge 0} - wv - hv + \alpha\mathbb{E}[f_{t+1,T}(v, Y_{t+1}, W_{t+1})].$$

Recall from Lemma 2 that $f_{t+1,T}(v, Y_{t+1}, W_{t+1})$ is concave in v for all $(Y_{t+1}, W_{t+1}) \in \mathcal{Y} \times \mathcal{W}$. Therefore, the objective function in (4.4) is concave in v. This, along with the definition of $\mathbf{v}^* = \{v_{t,T}^*(\cdot)\}$ immediately implies the following result.

Theorem 7. *The policy* $\pi^S(\mathbf{v}^*)$ *is an optimal policy for Problem* P_T*.*

To assess the relative impact of various factors on distressed selling, it is useful to compute quantities such as the total volume of distressed sales and the profit of the farmers under the optimal policy for P_T. However, due to the non-stationary and state-dependent nature of this policy, it is difficult to obtain closed-form expressions for these quantities. Therefore, we develop an approximation that simplifies our analysis to the extent that it enables obtaining these quantities in closed-form. In the real data that we will discuss later in Section 4.5, we make two observations: (1) The selling horizon T is large (typically around 150 days or more). (2) The range $(\overline{w} - \underline{w})/\underline{w}$ of the agent's price is small (typically around 10% or less); see Directorate of Economics and Statistics (2010b). Motivated by these observations, we consider an approximation $\hat{P}_\infty$ that differs from P_T in two respects: (1) $\hat{P}_\infty$ is an infinite-horizon problem, and (2) the agent's price is fixed at $\overline{w}$. That is, Problem $\hat{P}_\infty$ is that of maximizing

$$\mathbb{E}\left[\sum_{t=1}^{\infty} \alpha^{t-1}\left(Sq_t^G + \overline{w}q_t^A - hx_{t+1}\right)\right].$$

4.4 A Tractable Infinite-Horizon Approximation

In this section, we show that a threshold-based policy defined by a single constant is optimal for Problem $\hat{P}_\infty$ and obtain a closed-form expression for that constant threshold. Moreover, we show that the optimal profit in Problem $\hat{P}_\infty$ closely approximates the optimal profit in Problem P_T for realistic values of our model parameters. Thus, for the purpose of deriving insights, we use $\hat{P}_\infty$ as a proxy for the original problem P_T.

4.4.1 Optimal Policy for the Infinite Horizon Problem $\hat{P}_\infty$

We begin our analysis of $\hat{P}_\infty$ with a definition. A *constant sell-down-to* v policy, denoted by $\pi^{CS}(v)$, is defined by a constant number, v, as follows: It sells the maximum possible quantity to G in every period. If the inventory in the first period after selling to G exceeds v, then it sells enough to A to bring the inventory down to the level v. There are no sales to A beyond the first period. Mathematically, this policy is defined by the following relations:

$$q_t^G \;=\; \min(x_t, Y_t)\ \forall\ t, \quad q_1^A \;=\; (x_1 - Y_1 - v)^+, \text{ and } q_t^A = 0\ \forall\ t\ \geq\ 2.$$

We refer to v as the constant sell-down-to threshold. Next, we define a specific constant threshold, $\hat{v}$, and subsequently prove the optimality of the $\pi^{CS}(\hat{v})$ policy for problem $\hat{P}_\infty$.

Let $\tilde{Y}_1 = 0$ and $\tilde{Y}_t = (Y_2 + Y_3 + \ldots + Y_t)$ for all $t \geq 2$. Let us denote by $\hat{v}$ the solution to the following equation:

$$\sum_{t=2}^{\infty} \left(\alpha^{t-1} S - \frac{1-\alpha^{t-1}}{1-\alpha} h \right) P \left(\tilde{Y}_{t-1} \leq v < \tilde{Y}_t \right) = \overline{w}. \tag{4.5}$$

The left-hand side of (4.5) represents the expected marginal benefit of holding more than $v \geq 0$ units of inventory at the end of period 1; in particular, for any period $t = 2, 3, \ldots$, the marginal unit is sold to G in that period with probability $P(\tilde{Y}_{t-1} \leq v < \tilde{Y}_t)$ at profit equal to the discounted support price less the discounted sum of holding costs accrued over periods $1, 2, \ldots, t-1$. The right-hand side of (4.5) represents the guaranteed profit from selling the marginal unit to A in period 1.

Theorem 8. *The policy $\pi^{CS}(\hat{v})$ is an optimal policy for Problem $\hat{P}_\infty$.*

4.4.2 Comparative Statics

Given the optimal policy for Problem $\hat{P}_\infty$, we now explain the change in the threshold $\hat{v}$ with respect to the agent's price $\overline{w}$, the support price S, the holding cost h and the discount factor α. Notice that the amount of distressed sales is $(Q - Y_1 - \hat{v})^+$. Thus, when $Y_1 < Q - \hat{v}$, there is some amount of distressed selling. When this happens, the change in the amount of distressed sales due to the change in one of the parameters, $\overline{w}$, S, h or α, is exactly equal to the change in the value of $\hat{v}$ but in the reverse direction. The result below has the intuitive implication that a higher holding cost, a higher cost of capital (i.e., lower α), a lower support price or a more attractive price from the agent increases the quantity of distressed sales.

Theorem 9. *The threshold $\hat{v}$ is increasing in α and S, and decreasing in $\overline{w}$ and h.*

Next, we show that greater variability in the procurement capacities makes F worse off (i.e., reduces F's optimal profit). To capture this idea formally, we use the well known notion of convex ordering of random variables, whose definition we recall below.

Definition 1: (Shaked and Shanthikumar 2007) For any two random variables X and Y such that $\mathbb{E}[X] = \mathbb{E}[Y] < \infty$, X is said to be smaller in the convex order than Y (denoted by $X \leq_{cx} Y$) if the following statement holds: For every convex function $H : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[H(X)] \leq \mathbb{E}[H(Y)]$ if these expectations exist.

Theorem 10. *Let Y^A and Y^B be two random variables such that $Y^A \leq_{cx} Y^B$. Let $\{Y_t^A\}$ and $\{Y_t^B\}$ be two sequences of i.i.d. realizations of the random variables Y^A and Y^B, respectively. Then, for every $Q \geq 0$ and $y \in \mathcal{Y}$, $\hat{f}_\infty^A(Q, y) \geq \hat{f}_\infty^B(Q, y)$, where $\hat{f}_\infty^A$ and $\hat{f}_\infty^B$ denote the optimal profit functions of F when the procurement capacities are given by the sequences $\{Y_t^A\}$ and $\{Y_t^B\}$, respectively. Moreover, $\mathbb{E}[\hat{f}_\infty^A(Q, Y_1^A)] \geq \mathbb{E}[\hat{f}_\infty^B(Q, Y_1^B)]$.*

4.4.3 Closeness of the Approximate Problem $\hat{P}_\infty$ to the Original Problem P_T

In this sub-section, we answer the following question: How good an approximation of P_T is $\hat{P}_\infty$? Recall that $f_{1,T}(Q, y, w)$ is the optimal profit in P_T, starting from the state (Q, y, w). Let $\hat{f}_\infty(Q, y)$ denote the optimal profit in $\hat{P}_\infty$ starting from the state (Q, y) (recall that the agent's price is $\overline{w}$ in every period in $\hat{P}_\infty$). Also, let μ and σ respectively denote the mean and the standard deviation of the procurement capacity distribution ϕ. In Theorem 11 below, we bound the relative error $\frac{\hat{f}_\infty(Q,y) - f_{1,T}(Q,y,w)}{f_{1,T}(Q,y,w)}$ from above by a constant Δ. This development uses the seminal work on *distribution-free newsvendor bounds* of Scarf (1958) and Gallego and Moon (1993).

Theorem 11. *For every $T \in \mathbb{N}$, $Q > 0$, $y \in \mathcal{Y}$ and $w \in \mathcal{W}$, the ratio $\frac{\hat{f}_\infty(Q,y) - f_{1,T}(Q,y,w)}{f_{1,T}(Q,y,w)}$ is bounded from above by*

$$\Delta = \frac{\overline{w} - \underline{w}}{\underline{w}} + \frac{\alpha^T S}{\underline{w}} \frac{\left[(T-1)\sigma^2 + \left(\hat{v} - (T-1)\mu\right)^2\right]^{1/2} + \left[\hat{v} - (T-1)\mu\right]}{2\hat{v}}.$$

We check the performance of the bound Δ on real data. Although a more detailed description of our data will be provided in Section 4.5, a summary of the values of the model parameters that are used to compute this bound is as follows: $S = 10$, $T = 150$, $h = 2.217 \times 10^{-3}$, $\alpha \in [0.9986, 0.9971]$, $\mu = [300, 3000]$, $\overline{w} \in [8, 8.5]$ and $\underline{w} = \overline{w}/1.1$. Under exponentially-distributed procurement capacities (that fits the real data reasonably well; see Section 4.5 for details), the value of the bound, in the worst case, is 0.1161, thus illustrating the closeness of $\hat{P}_\infty$ to P_T.

4.4.4 Special Case of Exponentially Distributed Capacities: Closed Form Expressions

Let us now turn our attention to the special case of exponentially-distributed procurement capacities. This special case is important for several reasons: (1) As will be shown later in Section 4.5, the exponential distribution fits the real data reasonably well. (2) We can obtain closed-form expressions for several quantities of interest in Problem $\hat{P}_\infty$, e.g., the optimal profit and the loss in welfare of the farmers. (3) The exponential distribution results in the worst-case loss in welfare over all NBUE[4]-distributed procurement capacities.

The result below specifies the closed-form expressions for the constant sell-down-to threshold $\hat{v}$, the expected amount of distressed sales, and the optimal expected profit in $\hat{P}_\infty$, when the procurement capacities of G are exponentially-distributed. Later, in Sections 4.5 and 4.6, we will use these expressions to assess the performance of our model on real data and to suggest their use in policy-making.

Theorem 12. *Assume that the procurement capacities $\{Y_t\}$ are i.i.d. and exponentially distributed with mean $1/\lambda$. Then, the following statements hold:*

1. *The constant sell-down-to threshold is*

$$\hat{v} = \frac{1}{\lambda(1-\alpha)} \log\left(\frac{\alpha(1-\alpha)S + \alpha h}{(1-\alpha)\overline{w} + h}\right). \tag{4.6}$$

2. *The expected amount of distressed sales is*

$$\mathbb{E}_{Y_1}[(Q - Y_1 - \hat{v})^+] = (Q - \hat{v})^+ \; - \; \frac{1 - e^{-\lambda(Q-\hat{v})^+}}{\lambda}.$$

3. *The maximum expected discounted sum of profits is*

$$\begin{aligned} \mathbb{E}_{Y_1}[\hat{f}_\infty(Q, Y_1)] &= \hat{G}(Q) \quad \text{if } Q < \hat{v}, \qquad (4.7)\\ &= [(S-\overline{w})/\lambda][1 - e^{-\lambda(Q-\hat{v})}] + (Q-\hat{v})\overline{w} + \hat{G}(\hat{v})e^{-\lambda(Q-\hat{v})} \\ &\quad + [\alpha\hat{G}(\hat{v}) - h\hat{v}][1 - e^{-\lambda(Q-\hat{v})}] \quad \text{if } Q \geq \hat{v}, \qquad (4.8) \end{aligned}$$

and

$$\hat{G}(x) = \frac{[(1-\alpha)S + h][1 - e^{-(1-\alpha)\lambda x}]}{(1-\alpha)^2\lambda} \; - \; \frac{hx}{1-\alpha} \quad \forall\, x \geq 0. \tag{4.9}$$

Let $L(Q) = SQ - \mathbb{E}_{Y_1}[\hat{f}_\infty(Q, Y_1)]$ be the loss in welfare of the farmers due to distressed sales. The closed-form expressions in Theorem 12 can be useful in quickly quantifying the impact of an increase in procurement capacity or an increase in the discount factor (which can be achieved by making capital more affordable to farmers) on the loss in welfare; we will discuss this in more detail in Section 4.6.

Next, we show that exponential distribution provides the worst-case loss in welfare corresponding to any NBUE-distributed procurement capacities. First, we define the class of NBUE distributions and then, state our result formally.

[4]NBUE stands for New Better than Used in Expectation and is a commonly-used family of distributions that includes exponential, Weibull, and truncated normal distributions. In fact, the family of IFR distributions is a subset of the NBUE family.

Definition 2: (Shaked and Shanthikumar 2007) *A non-negative random variable X is NBUE if* $\mathbb{E}[X - t|X > t] \leq \mathbb{E}[X]$ *for all* $t \geq 0$.

Theorem 13. *Assume that the procurement capacities $\{Y_t\}$ are i.i.d. and follow an NBUE distribution with mean $1/\lambda$. Then, for every $Q \geq 0$, the loss in welfare is bounded from above by the value of $L(Q)$ for exponentially-distributed capacities with mean $1/\lambda$.*

We now proceed to examine the loss in welfare (i.e., $L(Q)$) more closely. In particular, we want to differentiate between (a) the impact of limited but certain procurement capacities and (b) the impact of uncertainty in the procurement capacities on $L(Q)$. Let the procurement capacity in every period be $1/\lambda$. In this case, it is easy to see that the optimal sell-down-to threshold is

$$v^D = \frac{1}{\lambda}\left\lfloor 1 + \frac{1}{\log(1/\alpha)} \log\left(\frac{\alpha(1-\alpha)S + \alpha h}{(1-\alpha)\overline{w} + h}\right)\right\rfloor$$

and the optimal profit is

$$F_\infty^D(Q) = \begin{cases} S/\lambda + \left(Q - 1/\lambda - v^D\right)\overline{w} + \sum_{t=1}^{\lambda v^D} \frac{1}{\lambda}\left(\alpha^t S - \frac{1-\alpha^t}{1-\alpha}h\right); & Q > 1/\lambda + v^D, \\ S/\lambda + \sum_{t=1}^{\hat{t}-1} \frac{1}{\lambda}\left(\alpha^t S - \frac{1-\alpha^t}{1-\alpha}h\right) + \left(Q - \frac{\hat{t}}{\lambda}\right)\left(\alpha^{\hat{t}} S - \frac{1-\alpha^{\hat{t}}}{1-\alpha}h\right); & 1/\lambda < Q \leq 1/\lambda + v^D, \\ SQ; & Q \leq \frac{1}{\lambda}, \end{cases}$$

where $\hat{t}$ is the number of periods required to sell $Q - 1/\lambda$ units to G at the rate $1/\lambda$, i.e., $\hat{t} = \lceil \lambda Q - 1 \rceil$. Define

$$L^D(Q) = SQ - F_\infty^D(Q)$$

as the loss in welfare due to limited but certain procurement capacities and

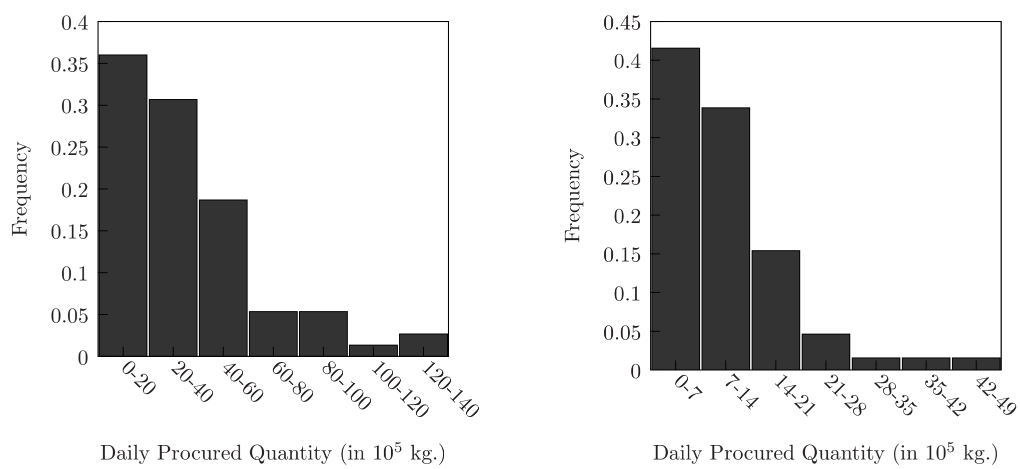
$$L^U(Q) = F_\infty^D(Q) - \mathbb{E}_{Y_1}\left[\hat{f}_\infty(Q, Y_1)\right]$$

as the loss in welfare due to the uncertainty in procurement capacities. Clearly, $L(Q) = L^D(Q) + L^U(Q)$. Using Theorem 10, we have $F_\infty^D(Q) \geq \mathbb{E}_{Y_1}[\hat{f}_\infty(Q, Y_1)]$ for all Q; thus, $L^U(Q) \geq 0$. We will use this partition of $L(Q)$ into $L^D(Q)$ and $L^U(Q)$ in Section 4.6 to examine the contribution of capacity uncertainty towards the total loss in welfare.

In the next section, we establish the credibility of our model and analysis by using real-world data on (a) the procurement of a major crop under a support-price program and (b) the other parameters we need – the total production by the farmers, the support price, the market price, the holding cost, and the discount factor. Finally, in Section 4.6, we illustrate how our analysis can be of help to policy-makers in taking decisions to improve the welfare of the farmers.

4.5 Real-World Data

Our aim in this section is to use real data to show that the predictions of our model are reasonable. We use real data on the procurement of rice – which, in its unmilled form, is referred to as *paddy* – in multiple districts by the government of the state of Odisha, India, under the central government's support-price program in 2010–11. The government's procurement from farmers takes place in each district, usually in its commercial capital. For illustration, we consider the procurement in the following five districts of Odisha: Balangir, Cuttack, Deogarh, Jajpur, and Khurda. For each district, we use the daily procured quantities of paddy by the state government during the five-month peak procurement season – from November 1, 2009 to March 31, 2010 – to estimate the distribution of the procurement capacities for the same period in 2010–11 (National Informatics Center 2010). This five-month interval, which follows the annual monsoon rains, is when the farmers sell most of their produce; about 85% of the total annual production of paddy is available by November 1 (International Rice Research Institute 2012). On average, the procurement occurs for four days in a week. Thus, a time period corresponds to $7/4 = 1.75$ days. Figure 4.1 illustrates this data for two districts.



(a) District Balangir, Odisha

(b) District Khurda, Odisha

Figure 4.1. Distribution of daily procured quantities of paddy during the peak procurement period in 2009–10.

Using the standard Kolmogorov-Smirnov one-sample test, we verify that the exponential distribution is a good fit for the per-period procurement capacities over the peak duration in each district; Table 4.1 provides the mean, μ, of the exponential distribution and the p-value for the K-S test corresponding to each district.

Table 4.1. The mean, μ, of the exponential distribution (1 unit = 1000 kgs.) and the p-value for the K-S test corresponding to each district.

Districts $\rightarrow$	Balangir	Cuttack	Deogarh	Jajpur	Khurda
Mean Per-Period Capacity, μ	3354.32	704.88	342.20	649.43	1037.24
p-value	0.3616	0.0700	0.2846	0.9387	0.4903

The other parameters required by our model were obtained for the 2010–11 procurement season from the following publicly-available documents: Directorate of Economics and Statistics (2010a), Planning Commission of India (2011), Directorate of Agriculture and Food Production (2010), The Times of India (2012a,b), and NDTV (2010). A summary of this data[5] is as follows: (1) The support price (S) = Rs. 10 per kg. (2) The per-period holding cost (h) = Rs. 2.217×10^{-3} per kg. (3) The annual interest rate ($\beta = 365(1-\alpha)/\alpha$) is in the range 30–60%. (4) The agent's price ($\overline{w}$) is in the range Rs. 8–8.5 per kg. (5) The district-wise total production of paddy by the farmers and the total procurement of paddy by the government for the five-month peak procurement period in 2010–11 are provided in Table 4.2.

Table 4.2. The total production of paddy by the farmers and the total procurement of paddy by the government for the five-month peak procurement period in 2010–11, respectively (1 unit = 10 million kgs.).

Districts $\rightarrow$	Balangir	Cuttack	Deogarh	Jajpur	Khurda
Total Quantity Produced, Q	50.488	31.136	3.793	21.976	21.446
Total Quantity Procured	23.547	5.087	1.921	3.793	7.414

4.5.1 Performance of the Approximation $\hat{P}_\infty$

We check the performance of our approximation $\hat{P}_\infty$ using the values outlined in the discussion above, namely, the support price S, the per-period holding cost h, the annual interest rate β, the agent's price $\overline{w}$, the per-period mean capacity μ, the total quantity Q produced by the farmers and the total quantity procured by the government.

For the annual interest rate β and the agent's price $\overline{w}$, only realistic ranges are available. Choosing the average values for both these parameters – namely, 45% for the annual interest rate and Rs. 8.25 per kg. for the agent's price – the percentage (absolute) deviations between

[5]Monetary values are expressed in the Indian currency – Rupees (Rs).

the predicted and the actual volume of distressed sales for the five districts are 11.27%, 1.92%, 42.39%, 7.92%, and, 5.72%, with an average deviation of 13.84%.

Given the limitations of our data and the admittedly stylized nature of our model, we adopt an alternative approach, which in our opinion is perhaps more practical and reasonable, to assess the predictions of our model: We first determine the values of the model parameters – namely, the annual interest rate β and the agent's price $\overline{w}$ – for which the predictions are accurate (i.e., the total predicted volume of distressed sales matches the actual value observed in the data). Then, we check if these imputed values of the parameters fall within their respective realistic ranges noted in publicly available reports. We now discuss this approach in further detail.

The typical ranges for the agent's price (about 15% to 20% below the support price) and the annual interest rate (about 30% to 60%) define the *"area of relevance"* in the space of these two parameters; this area is the shaded rectangle in Figure 4.2. For a district, consider an annual interest rate of, say, β. Let $\overline{w}_\beta$ be the corresponding price of the agent such that the total volume of distressed sales – as predicted by our model – matches the observed value of the distressed sales in that district; $\overline{w}_\beta$ can be calculated by using the closed-form expression for the optimal threshold $\hat{v}$ in Theorem 12 and the above-mentioned real-world values of the support price S, the per-period holding cost h, and the mean per-period capacity μ. By varying β, we obtain an imputed "*inference curve*" defined by the points $(\beta, \overline{w}_\beta)$ for which the predicted and the actual volume of distressed sales are equal; Figure 4.2 shows this inference curve for each of the five districts.

For the model prediction on distressed sales to be deemed reasonable for practical data, the inference curve for each district should intersect the area of relevance. In other words, for a reasonable reality check, the values of the parameters (the agent's price and the annual interest rate) that lead to accurate predictions should lie in the area of relevance. It is encouraging that there is, in fact, a substantial intersection of the inference curve and the area of relevance.

4.6 Use in Policy-Making

To policy-makers targeting a reduction in distressed sales, the direction in which each of its main determinants should change is clear: the procurement capacity should increase and its variability should decrease, the interest rates on the credit available to the farmers should decrease, and the cost the farmers incur in holding their produce should decrease. Given the

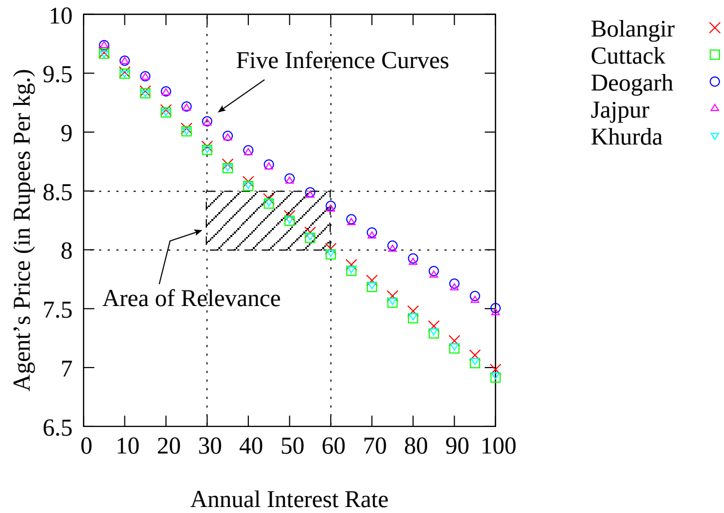
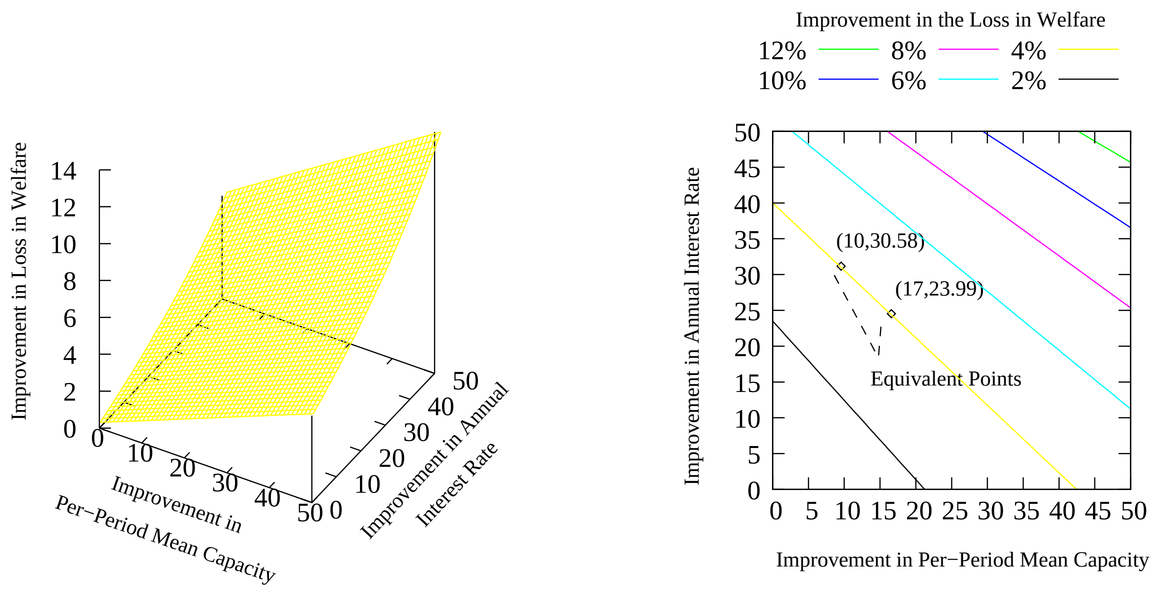


Figure 4.2. Performance of the approximation $\hat{P}_\infty$: The shaded region represents the "area of relevance" in the space of the agent's price and the annual interest rate. The collection of points represent imputed "inference curves" (one for each district).

scale of the agricultural sector in most developing countries, each of these improvements requires a significant capital investment. Further, these improvements require widely-differing infrastructural changes and the corresponding costs too can vary significantly across countries and regions; e.g., targeting an improved procurement mechanism poses challenges that are quite unlike those encountered in improving the nationwide storage infrastructure. Assuming that a modest budget is available for such improvements, it is then reasonable for policy-makers to aim for marginal improvements in one or more areas. In this context, an important use of our analysis can be in assessing the relative impacts of multiple and simultaneous improvements in the factors that affect distressed sales. We illustrate this below.

Consider, for example, improvements in the procurement capacity and the annual interest rate. Figure 4.3(a) uses Theorem 12 to show how the welfare of the farmers increases as a result of improvements in these two factors; the base case $(0, 0, 0)$ in this figure corresponds to the following realistic choice of the parameters: the total quantity produced, $Q = 250$ million kgs.; the support price, $S =$ Rs. 10 per kg.; the agent's price, $\overline{w}=$ Rs. 8.5 per kg.; the annual interest rate, $\beta = 25\%$; the per-period holding cost, $h =$ Rs. 2.217×10^{-3} per kg.; and the mean per-period procurement capacity, $\mu = 0.25$ million kgs. Figure 4.3(b) is the corresponding contour-plot representing isolines of the improvement in the loss in welfare of the farmers as a function of the mean procurement capacity and the annual interest rate. For a given improvement in the loss in welfare of the farmers, the contours provide various "equivalent" combinations of improvements in the mean capacity and the annual



(a) (b)

Figure 4.3. Improvement in the loss in welfare ($L(Q)$) of the farmers with respect to changes in the annual interest rate (β) and the mean per-period capacity (μ).

interest rate. For example, (i) a 10% increase in the mean per-period procurement capacity and a 30% reduction in the annual interest rate is equivalent to (ii) a 17% increase in that capacity and a 24% reduction in that interest rate, in the sense that both these combinations result in a 4% improvement in the loss in welfare of the farmers. The costs corresponding to these equivalent combinations, however, could be different. Thus, given the available budget, policy-makers can choose the best target improvement and the best combination to achieve it. Similarly, Figure 4.4(a) and 4.4(b) can be used to identify an ideal combination of improvements in the holding cost and the annual interest rate.

To further illustrate the discussion above, consider the following two improvement schemes that might be of interest to planners: (a) Procuring directly from small farmers in remote corners of the country by reaching out to them at the time of harvest, paying them immediately at the support price, and owning their produce. Thus, these farmers need not travel to the GPCs to sell their crops and the government assumes responsibility of the temporary storage and transport of the procured quantity to the GPCs; in India, for instance, the government is providing incentives to construct rural godowns for improving the storage infrastructure (Directorate of Marketing and Inspection 2015). (b) Increasing access to affordable credit for small farmers via improved budgetary allocations to village cooperatives and other rural

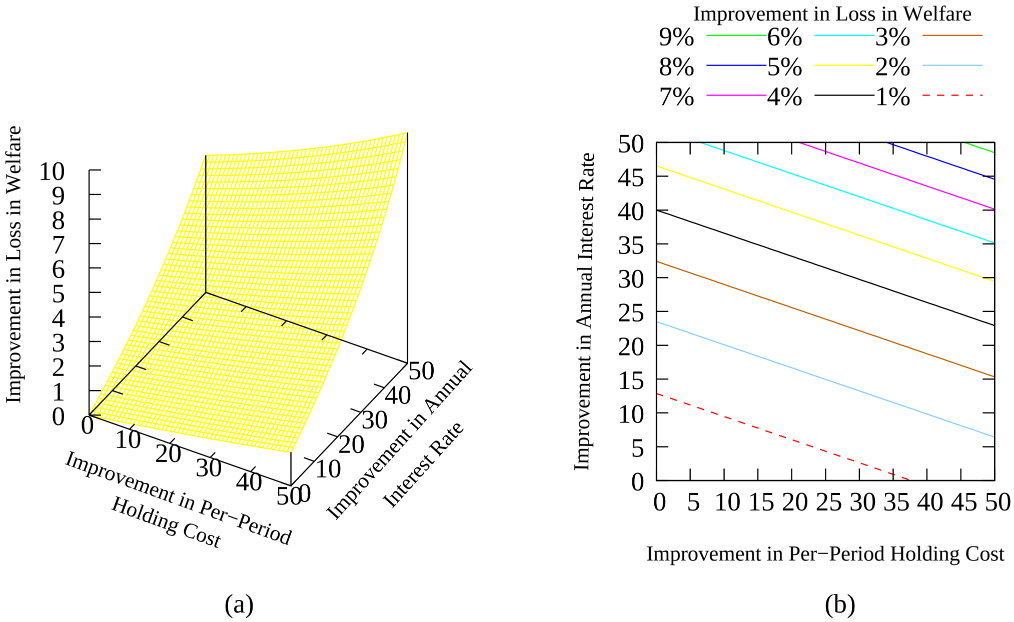
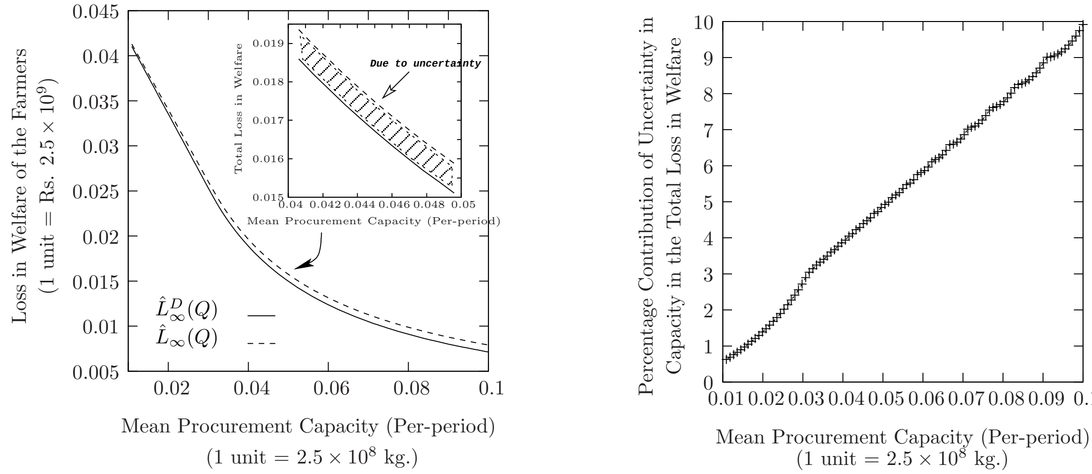


Figure 4.4. Improvement in the loss in welfare ($L(Q)$) of the farmers with respect to changes in the annual interest rate (β) and the holding cost (h).

financial institutions. The first scheme is essentially an attempt to increase the government's procurement capacity while the second can improve the discount rate for the farmers. Our analysis provides a simple tool for policy-makers to compare a variety of combinations of the marginal improvements through these two schemes.

Our analysis of the impact of variability in the procurement capacity at the GPCs on the loss in welfare of the farmers can also be useful in policy decisions. Recall our discussion in Section 4.4.4, where we divided the total loss in welfare ($L(Q)$) of the farmers into two components: (a) The loss in welfare due to limited but certain capacity ($L^D(Q)$) and (b) The loss in welfare due to uncertain capacity ($L^U(Q)$). Figure 4.5(a) shows the total loss in welfare and the loss in welfare due to limited but certain capacity, both as functions of the mean per-period capacity. Thus, the shaded-area between the two curves represents the loss in welfare of the farmers due to uncertain capacity. Figure 4.5(b) shows the percentage contribution of the uncertainty in capacity towards the total loss in welfare ($100 \times L^U(Q)/L(Q)$), as a function of the mean per-period capacity. When the capacity of the GPCs is small, it is clear that the loss in welfare of the farmers can be attributed almost entirely to the limited capacity. However, when the capacity is high, the uncertainty in capacity plays a significant role: approximately 10% of the loss in welfare is due to uncertainty in capacity. The ability



(a) (b)

Figure 4.5. Illustrating the differentiation between the loss in welfare of farmers due to limited but certain capacity ($L^D(Q)$) and uncertain capacity ($L^U(Q)$). This figure corresponds to the following choice of the parameters: the total quantity produced, Q = 250 million kg.; the support price, S = Rs. 10 per kg.; the agent's price, $\overline{w}$= Rs. 9.5 per kg., the interest rate, $\beta = 50\%$, and the per-period holding cost, h = Rs. 2.217×10^{-3} per kg.

to assess this impact via a simple, closed-form expression can be useful to policy-makers in making investments targeted at reducing the variability in the procurement capacity.

4.7 Conclusion and Future Research Directions

There is considerable interest among policymakers to alleviate the unfortunate practice of distressed selling by farmers in developing economies. Our attempt in this chapter has been to understand the role of the main factors – namely, the limited and uncertain procurement capacity of the government, the high holding cost incurred by the farmers, and the lack of affordable credit available to the farmers – that lead to this practice. To this end, our tractable model offers closed-form expressions for several quantities of interest; e.g., the total volume of distressed sales and the loss in welfare of the farmers. These, in turn, allow policymakers to compare the relative benefits of different infrastructural improvements that could potentially reduce distressed selling. Thus, given the status quo, policymakers can identify and target the most influential improvements.

Our work on distressed selling by farmers can be extended in several directions. In our model, we make several modeling assumptions that keep the analysis tractable and help us achieve closed-form expressions for several quantities. Here, we discuss several realistic scenarios under which such assumptions may not hold and warrant further analysis: (1) There are multiple (heterogenous) farmers and multiple GPCs; each farmer makes his own decision in response to the decisions made by the remaining farmers. (2) Farmers are risk-averse, (3) The procurement capacities of the GPCs are correlated over time.

APPENDIX A

PROOFS FOR CHAPTER 2

Proof of Claim 1: Let GM($\mathcal{N}$, f) denote the mechanism in Goel et al. (2012). Recall from our discussion in Section 2.3.1 on the connection between GM($\mathcal{N}$, f) and DM($\mathcal{N}$, f) that the two mechanisms are identical, except for the difference in the definitions of the price meters. Moreover, GM($\mathcal{N}$, f) is an efficient mechanism, i.e., it results in an allocation vector that solves the following optimization problem:

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \sum_{i=1}^{N} c_i Q_i \\ \text{s.t.} \quad & \mathbf{Q} \in P_f, \ \sum_{i=1}^{N} Q_i = f(\mathcal{N}). \end{aligned}$$

In both mechanisms, the allocations given to the suppliers depend only on the sequence in which the suppliers leave. Let $\Pi^{GM}(\mathbf{c}) = (\Pi_1^{GM}, \Pi_2^{GM}, \ldots, \Pi_N^{GM})$ be the sequence in which suppliers leave in GM($\mathcal{N}$, f), where Π_i^{GM} is the index of the supplier who is the i^{th} one to leave. Analogously, let $\Pi^{DM}(\mathbf{c}) = (\Pi_1^{DM}, \Pi_2^{DM}, \ldots, \Pi_N^{DM})$ be the sequence in which the suppliers leave in DM($\mathcal{N}$, f). Let $\mathbf{Q}(\Pi^{GM}(\mathbf{c}))$ be the allocation vector in GM($\mathcal{N}$, f) corresponding to the sequence $\Pi^{GM}(\mathbf{c})$ and let $\mathbf{Q}(\Pi^{DM}(\mathbf{c}))$ be the allocation vector in DM($\mathcal{N}$, f) corresponding to the sequence $\Pi^{DM}(\mathbf{c})$.

Goel et al. (2012) show that in GM($\mathcal{N}$, f), each supplier has a dominant strategy to exit when the price meter hits his unit cost. Thus, Π_i^{GM} is the index of the supplier with the $(N-i+1)^{st}$-lowest *unit cost*. They also show that the function $\hat{f}$ in Step 3 of GM($\mathcal{N}$, f) remains submodular and non-decreasing. Therefore, the function $\hat{f}$ in Step 3 of DM($\mathcal{N}$, f) also remains submodular and non-decreasing. Since $\hat{f}$ is non-decreasing, the incremental awards made in DM($\mathcal{N}$, f) are non-negative. Using this fact, it is easy to see that it is a dominant strategy in DM($\mathcal{N}$, f) for each supplier to exit when his own price meter hits his unit cost (see also the related discussion in Chapter 1). This observation and the definition of our price meters imply that Π_i^{DM} is the index of the supplier with the $(N-i+1)^{st}$-lowest *virtual-cost*. Hence, $\Pi^{DM}(\mathbf{c}) = \Pi^{GM}(\boldsymbol{\psi}(\mathbf{c}))$ for any $\mathbf{c}$. Using the facts that $\mathbf{Q}(\Pi^{GM}(\cdot))$

is an efficient allocation rule and $\Pi^{DM}(\cdot) = \Pi^{GM}(\boldsymbol{\psi}(\cdot))$, it follows that $\mathbf{Q}(\Pi^{DM}(\mathbf{c}))$ solves MR-PM($\mathbf{c}$). This completes the proof of Claim 1. ■

Proof of Claim 2: The total expected payment in $(\mathbf{Q}^*, \mathbf{M}^*)$ is $\sum_{i=1}^N \mathbb{E}_{c_i} m_i^*(c_i)$, where $m_i^*(c_i) = \mathbb{E}_{\mathbf{c}_{-i}}[M_i^*(\mathbf{c})]$ for all $i, c_i \in [\underline{c}_i, \bar{c}_i]$. We show that DM($\mathcal{N}$, f) achieves the same total expected payment. While DM($\mathcal{N}$, f) is not a direct mechanism (i.e., suppliers are not directly asked to report their costs), we know from the Revelation Principle that there exists a direct revelation (incentive compatible) mechanism $(\hat{\mathbf{Q}}, \hat{\mathbf{M}})$ that achieves the same outcome as DM($\mathcal{N}$, f). Since $(\hat{\mathbf{Q}}, \hat{\mathbf{M}})$ and $(\mathbf{Q}^*, \mathbf{M}^*)$ are both incentive compatible mechanisms with identical allocations (Claim 1), we know from the Revenue Equivalence Theorem; see, e.g., Proposition 5.2 of Krishna (2002), that $m_i^*(c_i) - \hat{m}_i(c_i) = m_i^*(\bar{c}_i) - \hat{m}_i(\bar{c}_i)$ for all $i, c_i \in [\underline{c}_i, \bar{c}_i]$, where $\hat{m}_i(t_i) = \mathbb{E}_{\mathbf{c}_{-i}}[\hat{M}_i(t_i, \mathbf{c}_{-i})]$ for any $t_i \in [\underline{c}_i, \bar{c}_i]$. Next, observe that $m_i^*(\bar{c}_i) = \bar{c}_i q_i^*(\bar{c}_i)$, where $q_i^*(\bar{c}_i) = \mathbb{E}_{\mathbf{c}_{-i}}[Q_i^*(\bar{c}_i, \mathbf{c}_{-i})]$. Finally, from the definition of DM($\mathcal{N}$, f), it is easy to see that $\hat{m}_i(\bar{c}_i) = \bar{c}_i \hat{q}_i(\bar{c}_i)$, where $\hat{q}_i(\bar{c}_i) = \mathbb{E}_{\mathbf{c}_{-i}}[\hat{Q}_i(\bar{c}_i, \mathbf{c}_{-i})]$. Since $\hat{\mathbf{Q}}(\cdot) = \mathbf{Q}^*(\cdot)$, we have $\hat{m}_i(c_i) = m_i^*(c_i)$ for all i and $c_i \in [\underline{c}_i, \bar{c}_i]$. Thus, the total expected payment in DM($\mathcal{N}$, f) is $\sum_{i=1}^N \mathbb{E}_{c_i} \hat{m}_i(c_i) = \sum_{i=1}^N \mathbb{E}_{c_i} m_i^*(c_i)$. ■

Proof of Claim 3: Consider an arbitrary set $\mathcal{N}$ of suppliers, cost vector $\mathbf{c}$, quantity Q, individual capacities $\boldsymbol{\Gamma}$ and group capacities $\boldsymbol{\eta}$. Let the suppliers be indexed such that $\psi_1(c_1) \leq \psi_2(c_2) \leq \ldots \leq \psi_N(c_N)$. We first prove a sequence of results in Claims 3-A to 3-C. These results will then be used to prove Claims 3-D and 3-E; these two claims will complete the proof of Claim 3.

Definition 1 (Schrijver 2003): A set function f on $\mathcal{N}$ is *non-decreasing* if for any subset $\mathcal{S} \subseteq \mathcal{N}$ and an element $s \in \mathcal{N} \setminus \mathcal{S}$, we have $f(\mathcal{S} \cup \{s\}) \geq f(\mathcal{S})$.

Definition 2 (Schrijver 2003): A set function f on $\mathcal{N}$ is *submodular* if and only if $f(\mathcal{S} \cup \{s\}) - f(\mathcal{S}) \geq f(\mathcal{S} \cup \{s, t\}) - f(\mathcal{S} \cup \{t\})$ for any subset $\mathcal{S} \subseteq \mathcal{N}$ and distinct $s, t \in \mathcal{N} \setminus \mathcal{S}$.

Claim 3-A: Define a set function g on $\mathcal{N}$ as follows:

$$g(\mathcal{S}) = \sum_{k=1}^{M} \min\left(\sum_{j \in \mathcal{S}_k} \Gamma_j, \eta_k\right) + \sum_{j \in \mathcal{S}^0} \Gamma_j, \ \forall \mathcal{S} \subseteq \mathcal{N}, \tag{A.1}$$

where $\mathcal{S}_k = \mathcal{S} \cap G_k$ for all k and $\mathcal{S}^0 = \mathcal{S} \cap G^0$. Then g is non-decreasing and submodular.

Proof of Claim 3-A: We use Definition 1 to show that g is non-decreasing. Consider an arbitrary subset $\hat{\mathcal{S}} \subseteq \mathcal{N}$ and an element $\hat{s} \in \mathcal{N} \setminus \hat{\mathcal{S}}$. Let $\hat{\mathcal{S}}_k = \hat{\mathcal{S}} \cap G_k$ for all $k \in \mathcal{M}$ and $\hat{\mathcal{S}}^0 = \hat{\mathcal{S}} \cap G^0$. We have the following cases:

- If $\hat{s} \in G_k$ for some $k \in \mathcal{M}$, then $g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}}) = \min\{\sum_{j \in \hat{\mathcal{S}}_k \cup \{\hat{s}\}} \Gamma_j, \eta_k\} - \min\{\sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j, \eta_k\}$, which takes one the following three possible values:

$$g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}}) = \begin{cases} \Gamma_{\hat{s}}, & \sum_{j \in \hat{\mathcal{S}}_k \cup \{\hat{s}\}} \Gamma_j < \eta_k, \\ \eta_k - \sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j, & \sum_{j \in \hat{\mathcal{S}}_k \cup \{\hat{s}\}} \Gamma_j \geq \eta_k > \sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j, \\ 0, & \eta_k \leq \sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j. \end{cases} \quad \text{(A.2)}$$

  Clearly, $g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}}) \geq 0$.

- If $\hat{s} \in G^0$, then $g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}}) = \Gamma_{\hat{s}} \geq 0$.

Thus, g is non-decreasing. We now use Definition 2 to show that g is submodular. Consider an arbitrary subset $\hat{\mathcal{S}} \subseteq \mathcal{N}$ and distinct $\hat{s}, \hat{t} \in \mathcal{N} \setminus \hat{\mathcal{S}}$. We have the following cases:

- Suppose that $\hat{s} \in G_k$ for some $k \in \mathcal{M}$. Consider the following scenarios:
  - If $\eta_k \leq \sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j$, then $g(\hat{\mathcal{S}} \cup \{\hat{s}, \hat{t}\}) - g(\hat{\mathcal{S}} \cup \{\hat{t}\}) = g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}}) = 0$.
  - If $\sum_{j \in \hat{\mathcal{S}}_k \cup \{\hat{s}\}} \Gamma_j \geq \eta_k > \sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j$, then $g(\hat{\mathcal{S}} \cup \{\hat{s}, \hat{t}\}) - g(\hat{\mathcal{S}} \cup \{\hat{t}\}) \leq \eta_k - \sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j = g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}})$.
  - If $\sum_{j \in \hat{\mathcal{S}}_k \cup \{\hat{s}\}} \Gamma_j < \eta_k$, then using the observation that $g(\mathcal{S} \cup \{s\}) - g(\mathcal{S}) \leq \Gamma_s$, for any $\mathcal{S} \subseteq \mathcal{N}$ and $s \in \mathcal{N} \setminus \mathcal{S}$, we have $g(\hat{\mathcal{S}} \cup \{\hat{s}, \hat{t}\}) - g(\hat{\mathcal{S}} \cup \{\hat{t}\}) \leq \Gamma_{\hat{s}} = g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}})$.

- If $\hat{s} \in G^0$, then $g(\hat{\mathcal{S}} \cup \{\hat{s}\}) - g(\hat{\mathcal{S}}) = g(\hat{\mathcal{S}} \cup \{\hat{s}, \hat{t}\}) - g(\hat{\mathcal{S}} \cup \{\hat{t}\}) = \Gamma_{\hat{s}}$.

The submodularity of the function g follows. □

Using the definition of the functions f_c in (2.5) and g in (A.1), we have $f_c(\mathcal{S}) = \min\{Q, g(\mathcal{S})\}$ for all $\mathcal{S} \subseteq \mathcal{N}$.

Claim 3-B: The function f_c is submodular and non-decreasing.
Proof of Claim 3-B: Using Claim 3-A, we know that g is submodular and non-decreasing. That the function f_c is submodular, given that g is submodular, is established in Schrijver (2003), Section 44.6e. It is easy to see that f_c is non-decreasing. Consider any two sets $\mathcal{T}$ and $\mathcal{S}$ such that $\mathcal{T} \subseteq \mathcal{S} \subseteq \mathcal{N}$. Since g is non-decreasing, we have $g(\mathcal{T}) \leq g(\mathcal{S})$. Also, by definition, $f_c(\mathcal{T}) \leq g(\mathcal{T})$ and $f_c(\mathcal{T}) \leq Q$. Thus, $f_c(\mathcal{T}) \leq \min\{Q, g(\mathcal{S})\} = f_c(\mathcal{S})$, establishing that f_c is non-decreasing. □

Claim 3-C: The feasible region of M-PM(f_c) is identical to that of M-CGP.

Proof of Claim 3-C: We will first show that the feasible region of M-PM(f_c) is a subset of that of M-CGP. Consider an arbitrary feasible solution $(\mathbf{Q}, \mathbf{M})$ to M-PM(f_c). To see that this solution is also feasible to M-CGP, we make the following observations:

- Since $(\mathbf{Q}, \mathbf{M})$ is feasible to M-PM(f_c), it must be incentive compatible and individually rational.

- Let g be the function defined in (A.1). The total quantity procured in $(\mathbf{Q}, \mathbf{M})$ is $f_c(\mathcal{N})$ units. Using $g(\mathcal{N}) \geq Q$ (Section 2.1), we have $f_c(\mathcal{N}) = Q$. Thus, the constraint $\sum_{i=1}^{N} Q_i = Q$ is satisfied.

- To show that the allocation vector $\mathbf{Q}$ satisfies the group capacity constraints, consider an arbitrary group $G_k, k \in \mathcal{M}$ and let $\hat{\mathcal{S}} = G_k$. Since $(\mathbf{Q}, \mathbf{M})$ is feasible to M-PM(f_c), we have $\sum_{i \in \hat{\mathcal{S}}} Q_i \leq f_c(\hat{\mathcal{S}}) = \min\{Q, g(\hat{\mathcal{S}})\} \leq g(\hat{\mathcal{S}})$. Using the definition of g in (A.1) and our assumption that the groups G_k, for all $k \in \mathcal{M}$ and G^0 are disjoint, we have $g(\hat{\mathcal{S}}) = \min\{\sum_{i \in G_k} \Gamma_i, \eta_k\} \leq \eta_k$. Thus, the constraint $\sum_{i \in G_k} Q_i \leq \eta_k$ is satisfied.

- To show that the allocation vector $\mathbf{Q}$ satisfies the individual capacity constraints, consider an arbitrary Supplier i. Since $(\mathbf{Q}, \mathbf{M})$ is feasible to M-PM(f_c), we have $Q_i \leq f_c(\{i\}) = \min\{Q, g(\{i\})\} \leq g(\{i\})$. If Supplier $i \in G^0$, then $Q_i \leq g(\{i\}) = \Gamma_i$. If Supplier $i \in G_k$ for some $k \in \mathcal{M}$, then $Q_i \leq g(\{1\}) = \min\{\Gamma_i, \eta_k\} \leq \Gamma_i$. Thus, the constraint $Q_i \leq \Gamma_i$ is satisfied.

We complete the proof of the claim by showing that the feasible region of M-CGP is a subset of that of M-PM(f_c). Consider an arbitrary feasible solution $(\mathbf{Q}, \mathbf{M})$ to M-CGP. To see that this solution is also feasible to M-PM(f_c), we first make the following observations: (1) $(\mathbf{Q}, \mathbf{M})$ is incentive compatible and individually rational, and (2) the constraint $\sum_{i=1}^{N} Q_i = f_c(\mathcal{N})$ is satisfied. Thus, it only remains to show that the allocation vector $\mathbf{Q}$ satisfies the feasibility constraints P_{f_c}. Suppose, to aim for a contradiction, that $\exists \hat{\mathcal{S}} \subseteq \mathcal{N}$ such that $\sum_{i \in \hat{\mathcal{S}}} Q_i > f_c(\hat{\mathcal{S}})$. By definition, $f_c(\hat{\mathcal{S}}) = \min\{Q, g(\hat{\mathcal{S}})\}$. Recall that $\hat{\mathcal{S}}_k = \hat{\mathcal{S}} \cap G_k$, for all $k \in \mathcal{M}$ and $\hat{\mathcal{S}}^0 = \hat{\mathcal{S}} \cap G^0$. If $f_c(\hat{\mathcal{S}}) = Q$, then we have a contradiction since the total units awarded in $\mathbf{Q}$ is Q. If $f_c(\hat{\mathcal{S}}) = g(\hat{\mathcal{S}})$, then $\sum_{j \in \hat{\mathcal{S}}} Q_j > f_c(\hat{\mathcal{S}})$ implies that $\sum_{k=1}^{M} \sum_{j \in \hat{\mathcal{S}}_k} Q_j + \sum_{j \in \hat{\mathcal{S}}^0} Q_j > \sum_{k=1}^{M} \min\left(\sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j, \eta_k\right) + \sum_{j \in \hat{\mathcal{S}}^0} \Gamma_j$. By assumption, the allocation vector $\mathbf{Q}$ satisfies the feasibility constraints imposed in M-CGP. Therefore, $Q_j \leq \Gamma_j$, for all $j \in \mathcal{N}$ and $\sum_{j \in G_k} Q_j \leq \eta_k$, for all $k \in \mathcal{M}$. This along with definitions of $\hat{\mathcal{S}}^0$ and $\hat{\mathcal{S}}_k$, for all $k \in \mathcal{M}$, together imply that $\sum_{j \in \hat{\mathcal{S}}_k} Q_j \leq \min\left(\sum_{j \in \hat{\mathcal{S}}_k} \Gamma_j, \eta_k\right)$, for all $k \in \mathcal{M}$

and $\sum_{j\in\hat{\mathcal{S}}^0} Q_j \leq \sum_{j\in\hat{\mathcal{S}}^0} \Gamma_j$. Thus, $\sum_{i\in\hat{\mathcal{S}}} Q_i \leq \sum_{k=1}^{M} \min\left(\sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j, \eta_k\right) + \sum_{j\in\hat{\mathcal{S}}^0} \Gamma_j$, again a contradiction. The claim follows. □

Claim 3-D: The mechanism DM($\mathcal{N}$, f_c) results in an optimal solution to M-CGP.
Proof of Claim 3-D: The proof of this claim follows immediately from Claim 3-C, the observation that problems M-PM(f_c) and M-CGP have identical objective functions (namely, minimizing the total expected payment given to the suppliers), and the optimality of DM($\mathcal{N}$, f_c) for problem M-PM(f_c) (Theorem 3). □

Let DM-MCGP($\mathcal{N}$, Q, $\mathbf{\Gamma}$, $\boldsymbol{\eta}$) be the descending mechanism in Section 2.1.2, where Q is the quantity that the buyer wants to procure from the suppliers in the set $\mathcal{N}$ with individual capacities $\mathbf{\Gamma}$ and group capacities $\boldsymbol{\eta}$.

Claim 3-E: The incremental awards made in DM($\mathcal{N}$, f_c) and DM-MCGP($\mathcal{N}$, Q, $\mathbf{\Gamma}$, $\boldsymbol{\eta}$) are identical.
Proof of Claim 3-E: We use induction on the number of suppliers N. For the base case of a single supplier, it is easy to see that the awards given to this supplier in DM($\mathcal{N}$, f_c) and DM-MCGP($\mathcal{N}$, Q, $\mathbf{\Gamma}$, $\boldsymbol{\eta}$) are $f_c(\{1\})$ and Q, respectively. Since $f_c(\{1\}) = \min\{Q, \Gamma_1\}$ and $\Gamma_1 \geq Q$, these awards are identical.

Consider now the case of $N \geq 2$ suppliers. In DM($\mathcal{N}$, f_c), setting $\hat{\mathcal{N}} = \mathcal{N}$, $\hat{f} = f_c$ and $Q_i = 0$ for all $i \in \hat{\mathcal{N}}$, the buyer makes an award of $\delta_i^1 = f_c(\mathcal{N}) - f_c(\mathcal{N} \setminus \{i\})$ to Supplier $i \in \mathcal{N}$. On the other hand, in DM-MCGP($\mathcal{N}$, Q, $\mathbf{\Gamma}$, $\boldsymbol{\eta}$), the award given to Supplier $i \in \mathcal{N}$ is $\delta_i^2 = \max\{0, NEED - TRCwS_i\}$, where $NEED = Q$ and $TRCwS_i = g(\mathcal{N} \setminus \{i\})$ with the function g defined in (A.1). Using the definition of f_c in (2.5) and the assumption $g(\mathcal{N}) \geq Q$ in Section 2.1, we have $f_c(\mathcal{N}) = NEED$ and $f_c(\mathcal{N} \setminus \{i\}) = \min\{Q, g(\mathcal{N} \setminus \{i\})\} = \min\{NEED, TRCwS_i\}$. Thus, it is easy to see that $\delta_i^1 = \delta_i^2$ for all $i \in \mathcal{N}$. After these awards are made in DM($\mathcal{N}$, f_c), the buyer updates $\hat{f}(\mathcal{S}) = f_c(\mathcal{S}) - Q(\mathcal{S})$, where $Q(\mathcal{S}) = \sum_{j\in\mathcal{S}} Q_j$, for all $\mathcal{S} \subseteq \mathcal{N}$. On the other hand, in DM-MCGP($\mathcal{N}$, Q, $\mathbf{\Gamma}$, $\boldsymbol{\eta}$), the buyer updates $NEED = Q - Q(\mathcal{N})$, $RC_i = \Gamma_i - Q_i$, for all $i \in \mathcal{N}$, and $RCG_l = \eta_l - Q(G_l)$, for all $l \in \mathcal{M}$. Since $\hat{f}(\mathcal{N}) = Q - Q(\mathcal{N}) = NEED$, these two mechanisms either both end (this happens when $NEED = 0$) or both continue. The proof of Claim 3-E follows trivially when both mechanisms end. We now consider the case when both mechanisms continue. In particular, in both mechanisms, the buyer reduces the price meters until a supplier leaves. From the proof of Claim 1 and the trivial identification of dominant strategies in Chapter 1, we know that the suppliers exit in the sequence of their virtual-costs in both DM($\mathcal{N}$, f_c) and DM-MCGP($\mathcal{N}$, Q, $\mathbf{\Gamma}$, $\boldsymbol{\eta}$). Thus, in both mechanisms, the updated set of suppliers $\hat{\mathcal{N}}$

is given by $\hat{\mathcal{N}} = \{1, 2, \ldots, N-1\}$. Define $\hat{\Gamma}_i = RC_i$, for all $i \in \mathcal{N}$ and $\hat{\eta}_k = RCG_k$, for all $k \in \mathcal{M}$. Let $\hat{\mathbf{\Gamma}} = (\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots, \hat{\Gamma}_N)$ and $\hat{\boldsymbol{\eta}} = (\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_M)$. We will complete the proof of Claim 3-E by showing that the incremental awards made in DM($\hat{\mathcal{N}}$, $\hat{f}$) and DM-MCGP($\hat{\mathcal{N}}$, $Q - Q(\mathcal{N})$, $\hat{\mathbf{\Gamma}}$, $\hat{\boldsymbol{\eta}}$) are identical.

Consider an arbitrary subset $\mathcal{S} \subseteq \mathcal{N}$. Recall that $\mathcal{S}_k = \mathcal{S} \cap G_k$, for all $k \in \mathcal{M}$, and $\mathcal{S}^0 = \mathcal{S} \cap G^0$. Define the set functions $\hat{g}$ and $\hat{f}_c$ on $\mathcal{N}$ as follows:

$$\hat{g}(\mathcal{S}) = \sum_{k=1}^{M} \min\left(\sum_{j \in \mathcal{S}_k} \hat{\Gamma}_j, \hat{\eta}_k\right) + \sum_{j \in \mathcal{S}^0} \hat{\Gamma}_j, \ \forall \mathcal{S} \subseteq \mathcal{N}.$$

$$\hat{f}_c(\mathcal{S}) = \min\left\{Q - Q(\mathcal{N}), \hat{g}(\mathcal{S})\right\}, \ \forall \mathcal{S} \subseteq \mathcal{N}.$$

Note that the function $\hat{f}_c$ satisfies (2.5) for the total procurement quantity $Q - Q(\mathcal{N})$, individual capacity vector $\hat{\mathbf{\Gamma}}$, and group capacity vector $\hat{\boldsymbol{\eta}}$. Moreover, using the fact that the mechanism DM($\mathcal{N}$, f_c) results in an optimal solution to M-CGP (Claim 3-D) and the observations that $\hat{g}(\hat{\mathcal{N}})$ is the total residual capacity of the set $\hat{\mathcal{N}}$ of suppliers, and the quantity $Q - Q(\mathcal{N})$ is the additional need of the buyer, we have $\hat{g}(\hat{\mathcal{N}}) \geq Q - Q(\mathcal{N})$. Consider the mechanism DM($\hat{\mathcal{N}}$, $\hat{f}_c$) with the set of suppliers $\hat{\mathcal{N}}$ and set function $\hat{f}_c$. By our induction hypothesis, the incremental awards made in DM($\hat{\mathcal{N}}$, $\hat{f}_c$) and DM-MCGP($\hat{\mathcal{N}}$, $Q - Q(\mathcal{N})$, $\hat{\mathbf{\Gamma}}$, $\hat{\boldsymbol{\eta}}$) are identical. To complete the proof, it remains to show that the incremental awards made in DM($\hat{\mathcal{N}}$, $\hat{f}$) and DM($\hat{\mathcal{N}}$, $\hat{f}_c$) are identical; in other words, that the functions $\hat{f}$ and $\hat{f}_c$ are identical.

Recall that the function $\hat{f}(\mathcal{S}) = f_c(\mathcal{S}) - Q(\mathcal{S})$, for all $\mathcal{S} \subseteq \mathcal{N}$. This can be re-written as $\hat{f}(\mathcal{S}) = \min\{Q - Q(\mathcal{S}), g(\mathcal{S}) - Q(\mathcal{S})\}$, for all $\mathcal{S} \subseteq \mathcal{N}$. By definition, $\hat{g}(\mathcal{N}) = g(\mathcal{N}) - Q(\mathcal{N})$. As a result, $\hat{f}_c(\mathcal{N}) = \min\{Q - Q(\mathcal{N}), g(\mathcal{N}) - Q(\mathcal{N})\} = \hat{f}(\mathcal{N}) = Q - Q(\mathcal{N})$. Consider now an arbitrary subset $\hat{\mathcal{S}} \subset \mathcal{N}$ and the following cases:

1. $g(\hat{\mathcal{S}}) \geq Q$:
   Consider an arbitrary Supplier $i \in \mathcal{N} \setminus \hat{\mathcal{S}}$. Since g is non-decreasing (Claim 3-A), $g(\mathcal{N} \setminus \{i\}) \geq g(\hat{\mathcal{S}}) \geq Q$. Thus, $Q_i = f_c(\mathcal{N}) - f_c(\mathcal{N} \setminus \{i\}) = 0$. As a result, $\hat{g}(\hat{\mathcal{S}}) = g(\hat{\mathcal{S}}) - Q(\hat{\mathcal{S}})$ and $Q(\mathcal{N}) = Q(\hat{\mathcal{S}})$; thus establishing $\hat{f}_c(\hat{\mathcal{S}}) = \hat{f}(\hat{\mathcal{S}})$.

2. $g(\hat{\mathcal{S}}) < Q$:
   In this case, $\hat{f}(\hat{\mathcal{S}}) = g(\hat{\mathcal{S}}) - Q(\hat{\mathcal{S}})$. To prove $\hat{f}(\hat{\mathcal{S}}) = \hat{f}_c(\hat{\mathcal{S}})$, we will show that $\hat{g}(\hat{\mathcal{S}}) = g(\hat{\mathcal{S}}) - Q(\hat{\mathcal{S}})$ and $Q - Q(\mathcal{N}) \geq \hat{g}(\hat{\mathcal{S}})$. Recall that $\hat{g}(\hat{\mathcal{S}})$ and $g(\hat{\mathcal{S}}) - Q(\hat{\mathcal{S}})$ are as follows:

$$\hat{g}(\hat{\mathcal{S}}) = \sum_{k=1}^{M} \min\left(\sum_{j \in \hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(G_k)\right) + \sum_{j \in \hat{\mathcal{S}}^0} \hat{\Gamma}_j,$$

$$g(\hat{\mathcal{S}}) - Q(\hat{\mathcal{S}}) = \sum_{k=1}^{M} \min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(\hat{\mathcal{S}}_k)\right) + \sum_{j\in\hat{\mathcal{S}}^0} \hat{\Gamma}_j.$$

We show that the equality $\min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(G_k)\right) = \min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(\hat{\mathcal{S}}_k)\right)$ holds, for any $k = 1, 2, \ldots, M$. Consider an arbitrary $k \in \mathcal{M}$ and the following cases:

- $\sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j \geq \eta_k$:
  Consider an arbitrary Supplier $i \in G_k \setminus \hat{\mathcal{S}}_k$. The award given to this supplier is $Q_i = f_c(\mathcal{N}) - f_c(\mathcal{N} \setminus \{i\})$. Since $\sum_{j\in G_k\setminus\{i\}} \Gamma_j \geq \sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j \geq \eta_k$, we have $g(\mathcal{N}\setminus\{i\}) = g(\mathcal{N}) \geq Q$. Thus, $f_c(\mathcal{N}\setminus\{i\}) = f_c(\mathcal{N}) = Q$; and hence $Q_i = 0$. Consequently, we have $\min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(G_k)\right) = \min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(\hat{\mathcal{S}}_k)\right) = \eta_k - Q(\hat{\mathcal{S}}_k)$.

- $\sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j < \eta_k$:
  We will first show that $Q(G_k \setminus \hat{\mathcal{S}}_k) \leq \eta_k - \sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j$. As a consequence of this result, we will have $\min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(G_k)\right) = \min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(\hat{\mathcal{S}}_k)\right) = \sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j$. To prove $Q(G_k\setminus\hat{\mathcal{S}}_k) \leq \eta_k - \sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j$, note that $Q_j = f_c(\mathcal{N}) - f_c(\mathcal{N}\setminus\{j\})$ for all $j \in G_k \setminus \hat{\mathcal{S}}_k$ and $g(\mathcal{N}) - g((\mathcal{N} \setminus G_k) \cup \hat{\mathcal{S}}_k) \leq \eta_k - \sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j$. Using the submodularity of the function f_c (Claim 3-B), we have $\sum_{j\in G_k\setminus\hat{\mathcal{S}}_k} [f_c(\mathcal{N}) - f_c(\mathcal{N} \setminus \{j\})] \leq f_c(\mathcal{N}) - f_c((\mathcal{N} \setminus G_k) \cup \hat{\mathcal{S}}_k)$. Moreover, by the definition of f_c, we have $f_c(\mathcal{N}) - f_c((\mathcal{N} \setminus G_k) \cup \hat{\mathcal{S}}_k) = Q - \min\{Q, g((\mathcal{N} \setminus G_k) \cup \hat{\mathcal{S}}_k)\}$. If $g((\mathcal{N} \setminus G_k) \cup \hat{\mathcal{S}}_k) \geq Q$, then $Q(G_k \setminus \hat{\mathcal{S}}_k) \leq 0 < \eta_k - \sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j$. Otherwise, $Q(G_k \setminus \hat{\mathcal{S}}_k) \leq Q - g((\mathcal{N} \setminus G_k) \cup S_k) \leq g(\mathcal{N}) - g((\mathcal{N} \setminus G_k) \cup S_k) \leq \eta_k - \sum_{j\in\hat{\mathcal{S}}_k} \Gamma_j$.

Thus, $\min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(G_k)\right) = \min\left(\sum_{j\in\hat{\mathcal{S}}_k} \hat{\Gamma}_j, \eta_k - Q(\hat{\mathcal{S}}_k)\right)$ for all k. As a result, $\hat{g}(\hat{\mathcal{S}}) = g(\hat{\mathcal{S}}) - Q(\hat{\mathcal{S}})$. We will now show that $Q - Q(\mathcal{N}) \geq \hat{g}(\hat{\mathcal{S}})$. From the proof of Claim 1, we know that $\hat{f}$ is non-decreasing. Thus, $\hat{f}(\mathcal{N}) \geq \hat{f}(\hat{\mathcal{S}}) = g(\hat{\mathcal{S}}) - Q(\hat{\mathcal{S}}) = \hat{g}(\hat{\mathcal{S}})$. Since $\hat{f}(\mathcal{N}) = Q - Q(\mathcal{N})$, we have $Q - Q(\mathcal{N}) \geq \hat{g}(\hat{\mathcal{S}})$; establishing $\hat{f}(\hat{\mathcal{S}}) = \hat{f}_c(\hat{\mathcal{S}})$.

This completes the proof of Claim 3-E. □

The proof of Claim 3 follows as a consequence of Claims 3-D and 3-E. ■

Proof of Claim 4: Consider an arbitrary set $\mathcal{N}$ of suppliers, cost vector $\mathbf{c}$, quantity Q, and the parameters a, b, N_L, and N_H, defined in Section 2.2. Let the suppliers be indexed such that $\psi_1(c_1) \leq \psi_2(c_2) \leq \ldots \leq \psi_N(c_N)$. Let DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) be the descending mechanism in Section 2.2.2. We first prove three preliminary results in Claims 4-A, 4-B, and 4-C. These results will then be used to prove Claims 4-D and 4-E; these two claims will complete the proof of Claim 4.

Claim 4-A: The set function f_b is non-decreasing and submodular.
Proof of Claim 4-A: Consider sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{N}$. Since $1 - a(L - |\mathcal{S}|)^+$ is a non-decreasing function of $|\mathcal{S}|$, we have $f_b(\mathcal{A}) = Q \min \{b|\mathcal{A}|, 1 - a(L - |\mathcal{A}|)^+\} \leq Q \min \{b|\mathcal{B}|, 1 - a(L - |\mathcal{B}|)^+\} = f_b(\mathcal{B})$; thus f_b is non-decreasing. Define the function $u : \mathcal{N} \rightarrow R_+$ as follows: $u(|\mathcal{S}|) = f_b(\mathcal{S})$ for all $\mathcal{S} \subseteq \mathcal{N}$. To prove that f_b is submodular, we use the following result: f_b is submodular if u is concave in $|\mathcal{S}|$; see Proposition 1.1 in Lovàsz (1983). The fact that $u(|\mathcal{S}|)$ is concave in $|\mathcal{S}|$ follows from the observations that $1 - a(L - |\mathcal{S}|)^+$ is concave in $|\mathcal{S}|$ and the minimum of a linear and a concave function is concave. □

Claim 4-B: The feasible region of M-PM(f_b) is a superset of the feasible region of M-BR.
Proof of Claim 4-B: Consider an arbitrary feasible solution $(\mathbf{Q}, \mathbf{M})$ to M-BR. To show that this solution is also feasible to M-PM(f_b), we make the following observations:

- Since $(\mathbf{Q}, \mathbf{M})$ is feasible to M-BR, it must be incentive compatible and individually rational.
- The total quantity procured in $(\mathbf{Q}, \mathbf{M})$ is Q units. Since $f_b(\mathcal{N}) = Q$, the constraint $\sum_{i=1}^{N} Q_i = f_b(\mathcal{N})$ is satisfied.
- To show that the allocation vector $\mathbf{Q}$ satisfies the feasibility constraints P_{f_b}, assume – to aim for a contradiction – that $\exists \hat{\mathcal{S}} \subseteq \mathcal{N}$ such that $\sum_{i \in \hat{\mathcal{S}}} Q_i > f_b(\hat{\mathcal{S}})$. We consider the following cases:
  - $b|\hat{\mathcal{S}}| \leq 1 - a(L - |\hat{\mathcal{S}}|)^+$:
    We have $f_b(\hat{\mathcal{S}}) = Qb|\hat{\mathcal{S}}|$ and thus, $\sum_{i \in \hat{\mathcal{S}}} Q_i > Qb|\hat{\mathcal{S}}|$. Since $(\mathbf{Q}, \mathbf{M})$ is feasible to M-BR, we have $Q_i \leq Qb$ for all $i \in \mathcal{N}$. This contradicts $\sum_{i \in \hat{\mathcal{S}}} Q_i > Qb|\hat{\mathcal{S}}|$.
  - $b|\hat{\mathcal{S}}| > 1 - a(L - |\hat{\mathcal{S}}|)^+$ and $|\hat{\mathcal{S}}| > L$:
    We have $f_b(\hat{\mathcal{S}}) = Q$ and thus, $\sum_{i \in \hat{\mathcal{S}}} Q_i > Q$. This contradicts the fact that the total quantity procured in $(\mathbf{Q}, \mathbf{M})$ is Q units.
  - $b|\hat{\mathcal{S}}| > 1 - a(L - |\hat{\mathcal{S}}|)^+$ and $|\hat{\mathcal{S}}| \leq L$:
    Let $\mathcal{W}$ be the set of suppliers who receive a strictly positive allocation in $(\mathbf{Q}, \mathbf{M})$. Let $\mathcal{W}_{-\hat{\mathcal{S}}} = \mathcal{W} \setminus (\mathcal{W} \cap \hat{\mathcal{S}})$. We have $f_b(\hat{\mathcal{S}}) = Q - Qa(L - |\hat{\mathcal{S}}|)$ and thus, $\sum_{i \in \hat{\mathcal{S}}} Q_i > Q - Qa(L - |\hat{\mathcal{S}}|)$. Since $(\mathbf{Q}, \mathbf{M})$ is feasible to M-BR, we have $\sum_{i \in \mathcal{W}_{-\hat{\mathcal{S}}}} Q_i \geq aQ|\mathcal{W}_{-\hat{\mathcal{S}}}|$. Also, using the definition of L in Section 2.2, we have $|\mathcal{W}_{-\hat{\mathcal{S}}}| \geq L - |\hat{\mathcal{S}}|$. Thus, $\sum_{i \in \mathcal{W}_{-\hat{\mathcal{S}}}} Q_i \geq aQ(L - |\hat{\mathcal{S}}|)$. Since $\sum_{i \in \mathcal{W}_{-\hat{\mathcal{S}}}} Q_i + \sum_{i \in \hat{\mathcal{S}}} Q_i = Q$, we have $\sum_{i \in \hat{\mathcal{S}}} Q_i \leq Q - aQ(L - |\hat{\mathcal{S}}|)$, a contradiction.

The result now follows. $\square$

Claim 4-C: Consider a submodular and non-decreasing set function f on the set $\mathcal{N}$. The mechanism DM($\mathcal{N}$, f) results in a cumulative allocation of $f(\{1,2,\ldots,i\}) - f(\{1,2,\ldots,i-1\})$ units to Supplier i, for all $i \in \mathcal{N}$.

Proof of Claim 4-C: We use induction on the number of suppliers N. For the base case of a single supplier (i.e., $N = 1$), it is trivial to see that the descending mechanism awards $f(\{1\}) - f(\emptyset)$ units to that supplier. Consider now the case when the number of suppliers is $N \geq 2$. After the initialization in Step 2, the award given to Supplier i in Step 3 of DM($\mathcal{N}$, f) is $\delta_i = f(\mathcal{N}) - f(\mathcal{N} \setminus \{i\})$; $i = 1,2,\ldots,N$. After these awards are made, the buyer updates $\hat{f}(\mathcal{S}) = f(\mathcal{S}) - \sum_{j \in \mathcal{S}} \delta_j$, for all $\mathcal{S} \subseteq \mathcal{N}$. Consider the following cases:

- $\hat{f}(\mathcal{N}) = 0$: In this case, the buyer concludes the auction. We show that $\delta_i = f(\{1,2,\ldots,i\}) - f(\{1,2,\ldots,i-1\})$ for all $i \in \mathcal{N}$. From the proof of Claim 1, we know that $\hat{f}$ is submodular and non-decreasing. This along with the facts that $f(\emptyset) = 0$ and $\hat{f}(\mathcal{N}) = 0$, imply that $\hat{f}(\{1,2,\ldots,i\}) = 0$; and hence, $f(\{1,2,\ldots,i\}) = \sum_{j=1}^{i} \delta_j$ for all i. Thus, $\delta_i = f(\{1,2,\ldots,i\}) - f(\{1,2,\ldots,i-1\})$, $\forall i$.

- $\hat{f}(\mathcal{N}) > 0$: In this case, the buyer continues the auction by reducing the price meters until a supplier exits. From the proof of Claim 1, we know that suppliers exit in the decreasing order of their virtual-costs. Thus, Supplier N is the first supplier who exits the auction. Using our induction hypothesis, the mechanism DM($\{1,2,\ldots,N-1\}$, $\hat{f}$) results in an allocation vector $\mathbf{\Delta} = (\Delta_1, \Delta_2, \ldots, \Delta_{N-1})$, with $\Delta_i = \hat{f}(\{1,2,\ldots,i\}) - \hat{f}(\{1,2,\ldots,i-1\})$ for all $i = 1,2,\ldots,N-1$. Define $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_N)$, where $Q_i = \delta_i + \Delta_i$ for all $i = 1,2,\ldots,N-1$ and $Q_N = \delta_N = f(\{1,2,\ldots,N\}) - f(\{1,2,\ldots,N-1\})$. To complete the proof of our claim, we note that DM($\mathcal{N}$, f) results in the allocation vector $\mathbf{Q}$ and show that $Q_i = f(\{1,2,\ldots,i\}) - f(\{1,2,\ldots,i-1\})$ for all $i = 1,2,\ldots,N-1$. Consider the allocation to an arbitrary Supplier i, $1 \leq i \leq N-1$:

$$\begin{aligned}
Q_i &= \delta_i + \Delta_i = \delta_i + \hat{f}(\{1,2,\ldots,i\}) - \hat{f}(\{1,2,\ldots,i-1\}), \\
&= \delta_i + f(\{1,2,\ldots,i\}) - \sum_{j=1}^{i} \delta_j - f(\{1,2,\ldots,i-1\}) + \sum_{j=1}^{i-1} \delta_j, \\
&= \delta_i + f(\{1,2,\ldots,i\}) - \delta_i - f(\{1,2,\ldots,i-1\}), \\
&= f(\{1,2,\ldots,i\}) - f(\{1,2,\ldots,i-1\}).
\end{aligned}$$

This completes our proof of Claim 4-C. $\square$

Claim 4-D: The descending mechanism DM($\mathcal{N}$, f_b) results in an optimal solution to M-BR.
Proof of Claim 4-D: We first show that DM($\mathcal{N}$, f_b) provides a feasible solution to M-BR. In other words, it satisfies the following constraints: (1) It is an incentive compatible and an individually rational mechanism. (2) The total allocation given to the suppliers is Q. (3) Each supplier either receives an allocation of 0 or an allocation in the range $[aQ, bQ]$. (4) The number of winning suppliers, i.e., those receiving strictly positive allocations, is between N_L and N_H. We make the following observations about the mechanism DM($\mathcal{N}$, f_b):

- By Claim 4-A, we know that f_b is submodular and non-decreasing. Thus, by Theorem 3, DM($\mathcal{N}$, f_b) is an optimal solution to M-PM(f_b). Therefore, it is incentive compatible and individually rational.

- Recall from Section 2.2 that L is the smallest value of $n \in \{N_L, N_L+1, \ldots, N_H\}$ such that $(aQ)n \leq Q \leq (bQ)n$. Also, $K = \lfloor \frac{1-aL}{b-a} \rfloor$ and $\beta = 1 - aL - (b-a)K$. This, along with the definition of f_b in (2.6), implies the following:

$$f_b(\mathcal{S}) = \begin{cases} bQ|\mathcal{S}|, & |\mathcal{S}| = 1, 2, \ldots, K, \\ Q - aQ(L - |\mathcal{S}|), & |\mathcal{S}| = K+1, K+2, \ldots, L, \\ Q, & |\mathcal{S}| = L+1, L+2, \ldots, N. \end{cases} \tag{A.3}$$

  Using Claim 4-C, the mechanism DM($\mathcal{N}$, f_b) results in an allocation vector $\mathbf{Q}$, where the total allocation given to Supplier i is $Q_i = f_b(\{1, 2, \ldots, i\}) - f_b(\{1, 2, \ldots, i-1\})$. Using (A.3), we have:

$$Q_i = \begin{cases} bQ, & i \in \{1, 2, \ldots, K\}, \\ aQ + \beta Q, & i = K+1, \\ aQ, & i \in \{K+2, K+3, \ldots, L\}, \\ 0, & i \in \{L+1, L+2, \ldots, N\}. \end{cases}$$

  That is, (i) each of the K lowest-virtual-cost suppliers receives the maximum allocation of bQ, (ii) the $(K+1)^{st}$ lowest-virtual-cost supplier receives an intermediate allocation, and (iii) each of the next $[L-(K+1)]$ lowest-virtual-cost suppliers receives the minimum allocation of aQ. It is easy to verify that the total allocation given to the suppliers equals Q. Moreover, the allocation vector $\mathbf{Q}$ satisfies the business rules specified by the parameters a, b, N_L, and N_H.

Thus, DM($\mathcal{N}$, f_b) results in a feasible solution to M-BR. This property, the optimality of DM($\mathcal{N}$, f_b) for problem M-PM(f_b) (Theorem 3), Claim 4-B, and the observation that problems M-PM(f_b) and M-BR have identical objective functions (namely, minimizing the total expected payment given to the suppliers), together imply that DM($\mathcal{N}$, f_b) is an optimal solution to M-BR. □

Claim 4-E: The incremental awards made in DM($\mathcal{N}$, f_b) and DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) are identical.

Proof of Claim 4-E: From the proof of Claim 1 and the trivial identification of dominant strategies in Chapter 1, we know that the suppliers exit in the decreasing order of their virtual costs in both DM($\mathcal{N}$, f_b) and DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H). This and the fact that the price meters defined in (2.3) and (2.4) are identical imply that the two mechanisms have identical sets of remaining suppliers. Let $\hat{\mathcal{N}}$ denote the set of remaining suppliers. We use induction on $\hat{\mathcal{N}}$ to show that the incremental awards made in DM($\mathcal{N}$, f_b) and DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) are identical. In both mechanisms, the buyer makes incremental awards at time $t = 0$ and at time instants when the suppliers exit. Thus, it is sufficient to consider $\hat{\mathcal{N}}$ only at these time instants.

Consider the time instant $t = 0$ with the set $\mathcal{N}$ of suppliers. At this time instant, the incremental award given to Supplier $i \in \mathcal{N}$ in DM($\mathcal{N}$, f_b) is $\delta_i^1 = f_b(\mathcal{N}) - f_b(\mathcal{N} \setminus \{i\})$. We show that these awards are identical to those given in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at that time instant. Recall from Section 2.2 that L is the smallest value of $n \in \{N_L, N_L{+}1, \ldots, N_H\}$ such that $(aQ)n \leq Q \leq (bQ)n$. Also, $K = \lfloor \frac{1-aL}{b-a} \rfloor$ and $\beta = 1 - aL - (b-a)K$. By the definitions of K and L, we have $K \leq L \leq N$. Consider the following cases:

- $L = N$

  We analyze this case under the following three subcases:

  - If $K = L$, then using the definition of f_b in (A.3), we have $f_b(\mathcal{N}) = bQK$ and $f_b(\mathcal{N} \setminus \{i\}) = bQ(K-1)$. Thus, $\delta_i^1 = f_b(\mathcal{N}) - f_b(\mathcal{N} \setminus \{i\}) = bQ$ for all $i \in \mathcal{N}$. These awards are identical to those given in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at time $t = 0$ with N suppliers satisfying $K = L = N$.
  - If $K = L - 1$, then using (A.3), we have $f_b(\mathcal{N}) = Q$ and $f_b(\mathcal{N} \setminus \{i\}) = bQK$. Thus, $\delta_i^1 = f_b(\mathcal{N}) - f_b(\mathcal{N} \setminus \{i\}) = Q - bQK$ for all $i \in \mathcal{N}$. Using the definition of β and $K + 1 = L$, we obtain $Q - bQK = aQ + \beta Q$. Thus, $\delta_i^1 = aQ + \beta Q$ for all $i \in \mathcal{N}$. These awards are identical to those given in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at time $t = 0$ with N suppliers satisfying $K + 1 = L = N$.

- ∘ If $K < L - 1$, then using (A.3), we have $f_b(\mathcal{N}) = Q$ and $f_b(\mathcal{N} \setminus \{i\}) = Q - aQ(L - L + 1) = Q - aQ$. Thus, $\delta_i^1 = f_b(\mathcal{N}) - f_b(\mathcal{N} \setminus \{i\}) = aQ$ for all $i \in \mathcal{N}$. These awards are identical to those given in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at time $t = 0$ with N suppliers satisfying $K + 1 < L = N$.

- $L < N$
  Again, using (A.3), we have $f_b(\mathcal{N}) = f_b(\mathcal{N} \setminus \{i\}) = Q$. Thus, $\delta_i^1 = f_b(\mathcal{N}) - f_b(\mathcal{N} \setminus \{i\}) = 0$ for all $i \in \mathcal{N}$. These awards are identical to those given in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at time $t = 0$ with N suppliers satisfying $L < N$.

This establishes our claim for the incremental awards made at time instant $t = 0$.

Consider a time instant t at which a supplier leaves in both mechanisms. Let $\hat{\mathcal{N}}$ be the set of remaining suppliers at that time instant. Assume, as our induction hypothesis, that the incremental awards made in DM($\mathcal{N}$, f_b) and DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) are identical at all time epochs less than t. We use this to show that the incremental awards made at time t in both mechanisms are identical. Let Q_i be the sum of incremental awards given to Supplier $i \in \mathcal{N}$ at all time epochs less than t. The incremental award given to Supplier $i \in \hat{\mathcal{N}}$ in DM($\mathcal{N}$, f_b) is $\delta_i^1 = \hat{f}(\hat{\mathcal{N}}) - \hat{f}(\hat{\mathcal{N}} \setminus \{i\})$. Let δ_i^2 be the incremental award made to Supplier $i \in \hat{\mathcal{N}}$ in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H). We will complete our proof by showing that $\delta_i^1 = \delta_i^2$ for all $i \in \hat{\mathcal{N}}$. Consider the following cases:

- $|\hat{\mathcal{N}}| \geq L + 1$:
  Recall that in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H), the buyer makes no award as long as there are more than L suppliers remaining in the auction. Moreover, by the induction hypothesis, the incremental awards made in DM($\mathcal{N}$, f_b) and DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at time epochs less than t are identical. This and the assumption that $|\hat{\mathcal{N}}| \geq L + 1$ imply that $Q_i = 0$ for all $i \in \mathcal{N}$. Moreover, since $|\hat{\mathcal{N}} \setminus \{i\}| \geq L$, we have $\hat{f}(\hat{\mathcal{N}}) = \hat{f}(\hat{\mathcal{N}} \setminus \{i\}) = Q$. Thus, $\delta_i^1 = 0$ for all $i \in \hat{\mathcal{N}}$. These awards are identical to those given in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at time t when $|\hat{\mathcal{N}}| \geq L + 1$.

- $|\hat{\mathcal{N}}| = L$:
  Again, we have $Q_i = 0$ for all $i \in \mathcal{N}$. Moreover, $\hat{f}(\hat{\mathcal{N}}) = Q$ and $\hat{f}(\hat{\mathcal{N}} \setminus \{i\}) = Q - aQ$ for all $i \in \hat{\mathcal{N}}$. Thus, $\delta_i^1 = aQ$ for all $i \in \hat{\mathcal{N}}$. Recall that in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H), the buyer makes an award of aQ units to each remaining supplier when there are exactly L suppliers. Thus, $\delta_i^2 = aQ$ for all $i \in \hat{\mathcal{N}}$.

- $K+2 \leq |\hat{\mathcal{N}}| < L$:
  In this case, we have $Q_i = aQ$ for all $i = 1, 2, \ldots, L$, and $Q_i = 0$ for all $i = L+1, L+2, \ldots, N$. Moreover, using the definition of K and the assumption that $K+2 \leq |\hat{\mathcal{N}}| < L$, we have $\hat{f}(\hat{\mathcal{N}}) = Q - aQ(L - |\hat{\mathcal{N}}|) - aQ|\hat{\mathcal{N}}| = Q - aQL$ and $\hat{f}(\hat{\mathcal{N}} \setminus \{i\}) = Q - aQ(L - |\hat{\mathcal{N}} \setminus \{i\}|) - aQ|\hat{\mathcal{N}} \setminus \{i\}| = Q - aQL$ for all $i \in \hat{\mathcal{N}}$. Thus, $\delta_i^1 = 0$ for all $i \in \hat{\mathcal{N}}$. These awards are identical to those given in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) at time t when $K+2 \leq |\hat{\mathcal{N}}| < L$.

- $|\hat{\mathcal{N}}| = K+1$:
  In this case, we have $Q_i = aQ$ for all $i = 1, 2, \ldots, L$, and $Q_i = 0$ for all $i = L+1, L+2, \ldots, N$. Moreover, $\hat{f}(\hat{\mathcal{N}}) = Q - aQL$ and $\hat{f}(\hat{\mathcal{N}} \setminus \{i\}) = bQ|\hat{\mathcal{N}} \setminus \{i\}| - aQ|\hat{\mathcal{N}} \setminus \{i\}|$ for all $i \in \hat{\mathcal{N}}$. Thus, $\delta_i^1 = Q - aQL - (b-a)Q|\hat{\mathcal{N}} \setminus \{i\}| = \beta Q$ for all $i \in \hat{\mathcal{N}}$. Recall that in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H), the buyer makes an incremental award of βQ units to each remaining supplier when there are exactly $K+1$ suppliers. Thus, $\delta_i^2 = \beta Q$ for all $i \in \hat{\mathcal{N}}$.

- $|\hat{\mathcal{N}}| = K$:
  In this case, we have $Q_i = aQ + \beta Q$ for all $i = 1, 2, \ldots, K+1$, $Q_i = aQ$ for all $i = K+2, K+3, \ldots, L$, and $Q_i = 0$ for all $i = L+1, L+2, \ldots, N$. Moreover, $\hat{f}(\hat{\mathcal{N}}) = bQ|\hat{\mathcal{N}}| - (aQ + \beta Q)|\hat{\mathcal{N}}|$ and $\hat{f}(\hat{\mathcal{N}} \setminus \{i\}) = bQ|\hat{\mathcal{N}} \setminus \{i\}| - (aQ + \beta Q)|\hat{\mathcal{N}} \setminus \{i\}|$ for all $i \in \hat{\mathcal{N}}$. Thus, $\delta_i^1 = bQ - (aQ + \beta Q)$ for all $i \in \hat{\mathcal{N}}$. Recall that in DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H), the buyer makes an incremental award of $bQ - (aQ + \beta Q)$ units to each remaining supplier when there are exactly K suppliers. Thus, $\delta_i^2 = bQ - (aQ + \beta Q)$ for all $i \in \hat{\mathcal{N}}$.

- $|\hat{\mathcal{N}}| < K$:
  In this case, we have $Q_i = bQ$ for all $i = 1, 2, \ldots, K$, $Q_{K+1} = aQ + \beta Q$, $Q_i = aQ$ for all $i = K+2, K+3, \ldots, L$, and $Q_i = 0$ for all $i = L+1, L+2, \ldots, N$. Moreover, $\hat{f}(\hat{\mathcal{N}}) = bQ|\hat{\mathcal{N}}| - bQ|\hat{\mathcal{N}}| = 0$ and $\sum_{i=1}^{N} Q_i = Q$. Since $\hat{f}(\hat{\mathcal{N}}) = 0$, the mechanism DM($\mathcal{N}$, f_b) does not give any further awards; thus, $\delta_i^1 = 0$ for all $i \in \hat{\mathcal{N}}$. Recalling that DM-MBR($\mathcal{N}$, Q, a, b, N_L, N_H) ends after procuring exactly Q units, we have $\delta_i^2 = 0$ for all $i \in \hat{\mathcal{N}}$.

We conclude that $\delta_i^1 = \delta_i^2$ for all $i \in \hat{\mathcal{N}}$. This completes the proof of Claim 4-E. □

Claims 4-D and 4-E complete the proof of Claim 4. ■

Proof of Claim 5: Consider an arbitrary set $\mathcal{N}$ of suppliers, cost vector $\mathbf{c}$, and a symmetric polymatroid P_f. Let the suppliers be indexed such that $\psi_1(c_1) \leq \psi_2(c_2) \leq \ldots \leq \psi_N(c_N)$. From the description of DMC($\mathcal{N}$, f) in Section 2.4, we know that the allocations given to the suppliers in that mechanism are identical to those in DM($\mathcal{N}$, f). Moreover, the fact that DM($\mathcal{N}$, f) results in an optimal solution to M-PM(f) (Theorem 3) and the observation that the feasible region of M-PM(f) and M-PMC are identical for the set function f defined in Section 2.4, imply that DMC($\mathcal{N}$, f) results in an allocation vector that is a feasible solution to Problem MR-PMC($\mathbf{c}$). By Claim 4-C, this allocation vector is $\mathbf{Q}^C$, where $Q_i^C = f(\{1, 2, \ldots, i\}) - f(\{1, 2, \ldots, i-1\}) = w(i) - w(i-1)$, for all $i \in \mathcal{N}$. We will complete the proof of Claim 5 by showing that $\mathbf{Q}^C$ is an optimal solution to Problem MR-PMC($\mathbf{c}$). To this end, we first prove two intermediate results in Claims 5-A and 5-B.

Consider an arbitrary Q'_N such that $w(N) - w(N-1) \leq Q'_N \leq w(1) - w(0)$. Define the function $w' : \mathcal{N} \rightarrow R_+$ as follows: $w'(k) = \min\{w(k), w(k+1) - Q'_N\}$ for any $k \leq N-1$. Since $w(\cdot)$ is non-decreasing and concave, $w'(\cdot)$ is also non-decreasing and concave. Define the set function f' as follows: $f'(\mathcal{S}) = w'(|\mathcal{S}|)$ for all subsets $\mathcal{S} \subseteq \mathcal{N} \setminus \{N\}$. The function f' is submodular and non-decreasing; see Lovàsz (1983), Proposition 1.1. Let $P_{f'}$ be the polymatroid associated with f'. Let $\mathbf{q} = (q_1, q_2, \ldots, q_{N-1})$. Define the optimization problem MRR-PMC(Q'_N,$\mathbf{c}$) as follows:

$$\begin{aligned} \min_{\mathbf{q}} \quad & \sum_{i=1}^{N-1} \psi_i(c_i) H(q_i) \\ \text{s.t.} \quad & \mathbf{q} \in P_{f'}, \ \sum_{i=1}^{N-1} q_i = f'(\mathcal{N} \setminus \{N\}). \end{aligned} \tag{MRR-PMC(Q'_N,$\mathbf{c}$)}$$

Let $\mathbf{q}^*(Q'_N)$ be an optimal solution to Problem MRR-PMC(Q'_N,$\mathbf{c}$). Let

$$\mathbf{Q}(Q'_N) = (q_1^*(Q'_N), q_2^*(Q'_N), \ldots, q_{N-1}^*(Q'_N), Q'_N).$$

Claim 5-A: The allocation vector $\mathbf{Q}(Q'_N)$ is a feasible solution to MR-PMC($\mathbf{c}$).
Proof of Claim 5-A: We first note that $\sum_{i=1}^{N} Q_i(Q'_N) = f(\mathcal{N})$. This follows from the definition of w' and the assumption that $Q'_N \geq w(N) - w(N-1)$. To show that $\mathbf{Q}(Q'_N)$ is a feasible solution to MR-PMC($\mathbf{c}$), assume – to aim for a contradiction – that $\exists \hat{\mathcal{S}} \subset \mathcal{N}$ such that $\sum_{j \in \hat{\mathcal{S}}} Q_j(Q'_N) > f(\hat{\mathcal{S}})$. We have the following cases:

- If Supplier N belongs to $\hat{\mathcal{S}}$, then $\sum_{j \in \hat{\mathcal{S}} \setminus \{N\}} Q_j(Q'_N) > f(\hat{\mathcal{S}}) - Q'_N \geq \min\{f(\hat{\mathcal{S}} \setminus \{N\}), f(\hat{\mathcal{S}}) - Q'_N\} = f'(\hat{\mathcal{S}} \setminus \{N\})$. Thus, $\sum_{j \in \hat{\mathcal{S}} \setminus \{N\}} q_j^*(Q'_N) > f'(\hat{\mathcal{S}} \setminus \{N\})$, a contradiction to the fact that $\mathbf{q}^*(Q'_N)$ is an optimal solution to MRR-PMC(Q'_N,$\mathbf{c}$).

- If Supplier N does not belong to $\hat{\mathcal{S}}$, then $\sum_{j\in\hat{\mathcal{S}}} Q_j(Q'_N) = \sum_{j\in\hat{\mathcal{S}}} q^*_j(Q'_N) > f(\hat{\mathcal{S}}) \geq \min\{f(\hat{\mathcal{S}}), f(\hat{\mathcal{S}}\cup\{N\}) - Q'_N\} = f'(\hat{\mathcal{S}})$, again a contradiction to the fact that $\mathbf{q}^*(Q'_N)$ is an optimal solution to MRR-PMC(Q'_N,$\mathbf{c}$).

Thus, $\mathbf{Q}(Q'_N)$ is a feasible solution to MR-PMC($\mathbf{c}$). $\square$

Claim 5-B: Consider an optimal solution $\mathbf{Q}'$ to problem MR-PMC($\mathbf{c}$). Then, $\mathbf{Q}(Q'_N)$ is also an optimal solution to MR-PMC($\mathbf{c}$).

Proof of Claim 5-B: Since $\mathbf{Q}'$ is a feasible solution to MR-PMC($\mathbf{c}$), we have $w(N)-(N-1) \leq Q'_N \leq w(1) - w(0)$. The inequality $Q'_N \leq w(1) - w(0)$ follows directly from the definition of the function w and the constraint $Q'_N \leq f(\{N\})$. To establish $Q'_N \geq w(N) - w(N-1)$, assume – to aim for a contradiction – that $Q'_N < w(N) - w(N-1)$. Since $\mathbf{Q}'$ is feasible to MR-PMC($\mathbf{c}$), we have $\sum_{i\in\mathcal{N}\setminus\{N\}} Q'_i \leq w(N-1)$. Therefore, $\sum_{i=1}^{N} Q'_i = \sum_{i\in\mathcal{N}\setminus\{N\}} Q'_i + Q'_N < w(N-1) + w(N) - w(N-1) = w(N)$. This contradicts the feasibility constraint $\sum_{i=1}^{N} Q_i = f(\mathcal{N})$ in MR-PMC($\mathbf{c}$). Therefore, $w(N) - w(N-1) \leq Q'_N \leq w(1) - w(0)$. As a consequence of Claim 5-A, $\mathbf{Q}(Q'_N)$ is feasible to MR-PMC($\mathbf{c}$). To show that it is optimal, we compare the objective values in MR-PMC($\mathbf{c}$) corresponding to the solutions $\mathbf{Q}'$ and $\mathbf{Q}(Q'_N)$. Using the definition of $\mathbf{Q}(Q'_N)$ and the fact that $\mathbf{q}^*(Q'_N)$ is optimal to MRR-PMC(Q'_N,$\mathbf{c}$), we have $\sum_{i=1}^{N} \psi_i(c_i)H\big(Q_i(Q'_N)\big) = \sum_{i=1}^{N-1} \psi_i(c_i)H\big(Q_i(Q'_N)\big) + \psi_N(c_N)H\big(Q'_N\big) \leq \sum_{i=1}^{N-1} \psi_i(c_i)H\big(Q'_i\big) + \psi_N(c_N)H\big(Q'_N\big)$. The optimality of $\mathbf{Q}(Q'_N)$ for MR-PMC($\mathbf{c}$) follows from this inequality and the fact that $\mathbf{Q}'$ is an optimal solution to MR-PMC($\mathbf{c}$). $\square$

We now use Claim 5-B to show that the allocation vector $\mathbf{Q}^C$ is an optimal solution to MR-PMC($\mathbf{c}$). For this, we use induction on the number of suppliers N. For the base case of a single supplier (i.e., $N = 1$), it is trivial to see that the solution $Q^C_1 = w(1)$ is optimal to MR-PMC($\mathbf{c}$).

Consider now the case when there are $N \geq 2$ suppliers. Let $\mathbf{Q}'$ be an optimal solution to MR-PMC($\mathbf{c}$). As shown in the proof of Claim 5-B, we have $w(N) - w(N-1) \leq Q'_N \leq w(1) - w(0)$. Moreover, we know that $\mathbf{Q}(Q'_N) = (q^*_1(Q'_N), q^*_2(Q'_N), \ldots, q^*_{N-1}(Q'_N), Q'_N)$ is also an optimal solution to MR-PMC($\mathbf{c}$). Using our induction hypothesis, the solution $\mathbf{q}^*(Q'_N)$ to Problem MRR-PMC(Q'_N,$\mathbf{c}$) can be obtained as follows: $q^*_i(Q'_N) = f'(\{1,2,\ldots,i\}) - f'(\{1,2,\ldots,i-1\}) = w'(i) - w'(i-1)$ for all $i = 1,2,\ldots,N-1$.

Define an index $2 \leq k \leq N$ such that $w(k) - w(k-1) \leq Q'_N \leq w(k-1) - w(k-2)$. If there are multiple indices satisfying this condition, define k to be the highest such index. Using the definition of the index k and the fact that $w(\cdot)$ is concave, we have $w'(i) = w(i)$ for all $i \leq k-2$ and $w'(i) = w(i+1) - Q'_N$ for all $i \geq k-1$. As a result, $q^*_i(Q'_N)$ is as follows:

$$q_i^*(Q_N') = \begin{cases} w(i) - w(i-1), & i \leq k-2, \\ w(k) - Q_N' - w(k-2), & i = k-1, \\ w(i+1) - w(i), & k \leq i \leq N-1. \end{cases} \tag{A.4}$$

Let Δ denote the difference in the objective values in MR-PMC($\mathbf{c}$) corresponding to the solutions $\mathbf{Q}(Q_N')$ and $\mathbf{Q}^C$. Using (A.4), this difference can be written as:

$$\begin{aligned}
\Delta &= \sum_{i=1}^{N} \psi_i(c_i) H\Big(Q_i(Q_N')\Big) - \sum_{i=1}^{N} \psi_i(c_i) H\Big(Q_i^C\Big), \\
&= \psi_{k-1}(c_{k-1})\Big[H\Big(w(k) - Q_N' - w(k-2)\Big) - H\Big(w(k-1) - w(k-2)\Big)\Big] + \\
&\quad \sum_{i=k}^{N-1} \psi_i(c_i)\Big[H\Big(w(i+1) - w(i)\Big) - H\Big(w(i) - w(i-1)\Big)\Big] + \\
&\quad \psi_N(c_N)\Big[H\Big(Q_N'\Big) - H\Big(w(N) - w(N-1)\Big)\Big], \\
&\geq \psi_{k-1}(c_{k-1})\Big\{\Big[H\Big(w(k) - Q_N' - w(k-2)\Big) - H\Big(w(k-1) - w(k-2)\Big)\Big] + \\
&\quad \sum_{i=k}^{N-1} \Big[H\Big(w(i+1) - w(i)\Big) - H\Big(w(i) - w(i-1)\Big)\Big] + \\
&\quad \Big[H\Big(Q_N'\Big) - H\Big(w(N) - w(N-1)\Big)\Big]\Big\}, \\
&= \psi_{k-1}(c_{k-1})\Big\{H\Big(w(k) - Q_N' - w(k-2)\Big) - H\Big(w(k-1) - w(k-2)\Big) - \\
&\quad H\Big(w(k) - w(k-1)\Big) + H\Big(Q_N'\Big)\Big\}, \\
&= \psi_{k-1}(c_{k-1})\Big\{\Big[H\Big(Q_N'\Big) - H\Big(w(k) - w(k-1)\Big)\Big] - \\
&\quad \Big[H\Big(w(k-1) - w(k-2)\Big) - H\Big(w(k) - Q_N' - w(k-2)\Big)\Big]\Big\}, \\
&\geq 0.
\end{aligned}$$

The last inequality follows from our assumption that $H(\cdot)$ is concave and the fact that $Q_N' \leq w(k-1) - w(k-2)$. This inequality and the facts that $\mathbf{Q}^C$ is feasible to MR-PMC($\mathbf{c}$) and $\mathbf{Q}(Q_N')$ is optimal to MR-PMC($\mathbf{c}$), imply that $\mathbf{Q}^C$ is optimal to MR-PMC($\mathbf{c}$). This completes the proof of Claim 5. ■

APPENDIX B

PROOFS FOR CHAPTER 4

Proof of Lemma 2: The proof that $f_{t,T}(x,y,w)$ is increasing in x is trivial since, for all $x' \geq x$, we have

$$f_{t,T}(x',y,w) - f_{t,T}(x,y,w) \geq w(x'-x) \ \geq \ 0.$$

This is because, when we have x' units at the beginning of period t, we can follow the optimal policy as if we have only x units and sell the excess of $(x'-x)$ units to A in period t.

The proof of concavity is by induction. By definition, $f_{T+1,T}(x,y,w) = 0$ for all (x,y,w). Now, we proceed inductively. Assume that for some $t \in \{1,2,\ldots,T\}$, $f_{t+1,T}(x,y,w)$ is a concave function of x for all (y,w). It only remains to show that $f_{t,T}(x,y,w)$ is concave with respect to x. To do this, let us examine the definition of $f_{t,T}(x,y,w)$ in (4.1). Since $f_{t+1,T}(x,y,w)$ is concave in x and $(x-q^G-q^A)$ is a linear function of (x,q^G,q^A), we know that $f_{t+1,T}(x-q^G-q^A, Y_{t+1}, W_{t+1})$ is jointly concave in (x,q^G,q^A). The expectation and summation operations preserve concavity. Thus,

$$Sq^G + wq^A - h(x-q^G-q^A) \ + \ \alpha\mathbb{E}\left[f_{t+1,T}(x-q^G-q^A, Y_{t+1}, W_{t+1})\right]$$

is also jointly concave in (x,q^G,q^A). Then, the desired concavity of $f_{t,T}(x,y,w)$ in x follows immediately from (4.1) and a standard result on the preservation of concavity by maximization (see Theorem A.4, Porteus 2002).

To prove the last statement, it is sufficient to show that for all $x' \leq x$,

$$f_{t,T}(x,y,w) - f_{t,T}(x',y,w) \leq S(x-x')$$

which is equivalent to the statement that

$$f_{t,T}(x',y,w) \geq f_{t,T}(x,y,w) - S(x-x'). \tag{B.1}$$

To see this, we consider two systems, one starting from the state (x,y,w) in period t while another starts from the state (x',y,w) in that same period. Both systems experience the same capacity and agent price realizations in all periods. Let policy π achieve the optimal expected profit $f_{t,T}(x,y,w)$. We now define a feasible (but not necessarily optimal) policy

π' operated on the system which starts from the state (x', y, w). For every realization of capacities and agent prices in periods $t+1, t+2, \ldots, T$, policy π' does not sell any unit to G and to A until policy π has sold $(x - x')$ units; at that instant, both systems have the same amount of inventory – that is both policies *couple*; from then on, policy π' mimics π in the sense that it uses the same selling quantities as π. It should now be clear that for every realization of capacities and agent prices, the only difference in the profits obtained by π and π' is due to the profit associated with the $(x - x')$ units sold by π before π' couples with π. Since the agent's price is less than S with probability 1 in every period, it is obvious that this difference can be at most $S(x - x')$. Thus, the expected profit under π' from state (x', y, w) in period t exceeds $f_{t,T}(x, y, w) - S(x - x')$, which implies the desired result in (B.1) since $f_{t,T}(x', y, w)$ is the optimal expected profit from the state (x', y, w) in period t. ■

Proof of Lemma 3: Let $Y_t = y$, $W_t = w$ and $x_t = x$. Consider the maximization problem in (4.1). Let $q = q^G + q^A$. For any fixed q, the derivative of the objective with respect to q^G can be computed using the relation $q^A = q - q^G$. This derivative equals $S - w$ which is strictly positive by our assumption on the agent's price. This implies that it is optimal to have $q^A > 0$ only if q^G hits the upper bound imposed by the procurement capacity, y; that is, $q^G = y$ and $x > y$. In other words, every optimal solution falls into one (or both) of the following two cases: (a) $q^G = y$ or (b) $q^A = 0$. The desired result is automatically satisfied in case (a).

To establish the desired result in case (b), we need to show that, in this case, $q^G = \min(x, y)$. To show this, let us consider the maximization problem in (4.1) with the additional constraint that $q^A = 0$ (i.e., solutions corresponding to case (b)). That is, consider

$$\begin{aligned} \max_{q^G} \quad & Sq^G - h(x - q^G) \ + \ \alpha\mathbb{E}\left[f_{t+1,T}(x - q^G, Y_{t+1}, W_{t+1})\right] \\ \text{s.t.} \quad & 0 \le q^G \le y, \quad q^G \le x \ . \end{aligned}$$

Differentiating the objective function above with respect to q^G and using the fact that the derivative of $f_{t+1,T}$ with respect to its first argument is smaller than S (from Lemma 2), we find that the objective is a strictly increasing function of q^G. Therefore, the optimal solution is $q^G = \min(x, y)$, thus proving the desired result. ■

Proof of Theorem 8: By definition, $\hat{P}_\infty$ is an infinite-horizon version of P_T in which the agent's price is constant. Moreover, the government's support price and the distributions of the government's procurement capacities are also stationary. Thus, standard DP convergence arguments (see Hernandez-Lerma and Lassere 1996 for details) along with the optimality of a sell-down-to policy for P_T (Theorem 7) can be used to show the optimality of a constant sell-down-to policy for $\hat{P}_\infty$. Let us now turn our attention to the issue of determining the optimal constant sell-down-to threshold.

First, observe that, starting from period 2 onwards, all constant sell-down-to policies are identical. This is because every policy in this class sells the maximum possible quantity to G in every period and zero to A in every period beyond the first. Thus, when the policy $\pi^{CS}(v)$ is used, the expected present value in period 2 of the profits received by F over periods $2, 3, \ldots$, depends on v only through the inventory on hand at the end of period 1. Let $\hat{G}(z)$ be this value when the inventory on hand at the end of period 1 is z. Then, we have

$$\hat{G}(z) = \mathbb{E}\left[\sum_{t=2}^{\infty} \alpha^{t-2} S \min\left(\left(z - \tilde{Y}_{t-1}\right)^+,\ Y_t\right) - \sum_{t=2}^{\infty} \alpha^{t-2}\left(z - \tilde{Y}_t\right)^+ h\right]. \tag{B.2}$$

Let $\hat{V}_\infty(Q; v; y)$ be the expected discounted sum of profits in $\hat{P}_\infty$ when we start from the state (Q, y) and the policy $\pi^{CS}(v)$ is used. That is,

$$\begin{aligned}\hat{V}_\infty(Q; v; y) &= SQ \text{ if } y \geq Q,\\ &= Sy + \overline{w}(Q - y - v)^+ - h\min(Q - y, v) + \alpha\hat{G}(\min(Q - y, v)) \text{ if } y \leq Q.\end{aligned}$$

To find the optimal value of v (i.e., $\arg\max_{v\geq 0}\hat{V}_\infty(Q; v; y)$), we observe that v has no effect on the profit if $y \geq Q$ because the entire inventory is sold to G in period 1. Thus, it only remains to study the optimal value of v when $y < Q$. That is,

$$\max_{v\geq 0} \hat{H}(Q - y, v)\ , \quad \text{where } \hat{H}(x, v) := \overline{w}(x - v)^+ - h\min(x, v) + \alpha\hat{G}(\min(x, v)). \tag{B.3}$$

To prove the desired result that $\hat{v}$ is the optimal constant sell-down-to threshold, we see from (B.3) that it suffices to show that $\hat{v}$ is a maximizer of $\hat{H}(x, v)$ for any $x \geq 0$. Now, notice that for any given $x \geq 0$, the function $\hat{H}(x, v)$ is constant with respect to v for all $v \geq x$. Thus, to maximize this function, it is sufficient to consider $v \in [0, x]$, in which case we have $\hat{H}(x, v) = \overline{w}(x - v) - hv + \alpha\hat{G}(v)$. Therefore, we complete our proof by showing that $\hat{G}(v)$ is a concave function and $\hat{v}$ maximizes $-(\overline{w} + h)v + \alpha\hat{G}(v)$.

Using (B.2) and standard manipulations, we can rewrite $\hat{G}(v)$ as follows:

$$\hat{G}(v) = \sum_{t=2}^{\infty} \alpha^{t-2}\left((1 - \alpha)S + h\right)\mathbb{E}\left[\min\left(v, \tilde{Y}_t\right)\right] - \frac{vh}{1 - \alpha}. \tag{B.4}$$

Since $\min(v, \tilde{Y}_t)$ is a concave function of v for every realization of $\tilde{Y}_t$ and concavity is preserved under expectation and addition, we see from (B.4) that $\hat{G}(v)$ is a concave function. In fact, with our assumption that the density function ϕ is strictly positive, we can show that $\hat{G}(v)$ is strictly concave. Thus, $-(\overline{w}+h)v + \alpha\hat{G}(v)$ is also strictly concave. Differentiating $\hat{G}(v)$ using (B.4), we see that

$$\frac{d}{dv}[-(\overline{w}+h)v + \alpha\hat{G}(v)] = -(\overline{w}+h) + \sum_{t=2}^{\infty} \alpha^{t-1}\left((1-\alpha)S + h\right) p(\hat{Y}_t > v) - \frac{\alpha h}{1-\alpha}.$$

With some algebra, this equation can be rewritten as

$$\frac{d}{dv}[-(\overline{w}+h)v + \alpha\hat{G}(v)] = -\overline{w} + \sum_{t=2}^{\infty} \left(\alpha^{t-1}S - \frac{1-\alpha^{t-1}}{1-\alpha}h\right) P\left(\tilde{Y}_{t-1} \le v < \tilde{Y}_t\right).$$

From the definition of $\hat{v}$, we see that this derivative vanishes at $v = \hat{v}$. Therefore, $\hat{v}$ maximizes the function $-(\overline{w}+h)v + \alpha\hat{G}(v)$; in fact, it is the unique maximizer since that function is strictly concave. This completes the proof. ∎

Proof of Theorem 9: Recall from the proof of Theorem 8 that $\hat{v}$ maximizes $-(\overline{w}+h)v + \alpha\hat{G}(v)$. Using the form of $\hat{G}(v)$ derived in (B.4), the optimal threshold $\hat{v}$ can also be obtained by solving

$$\sum_{t=2}^{\infty}(1-\alpha)\alpha^{t-1}P(\tilde{Y}_t > v) = \frac{\overline{w}(1-\alpha)+h}{S(1-\alpha)+h}.$$

Using the fact that $P(\tilde{Y}_t > v)$ is decreasing in v, it is evident from the above equation that $\hat{v}$ is increasing in S and decreasing in $\overline{w}$ and h.

The proof of the claim that $\hat{v}$ is increasing in α is quite involved. We first consider a finite-horizon version (with T periods) of $\hat{P}_\infty$ – we refer to this as Problem $\hat{P}_T$. It is identical to Problem P_T except that the agent's price W_t in any period t is deterministic and equal to $\overline{w}$. Thus, we can apply Theorem 7 to conclude that there exist constants $\{\hat{v}_{t,T}\}$ such that the following policy is optimal for $\hat{P}_T$: Sell the maximum possible quantity to G in every period and sell to A to bring the inventory down to the level $\hat{v}_{t,T}$ (if inventory exceeds that level) in every period t. Moreover, we know from the proof of Theorem 8 (namely, the dynamic programming convergence arguments referred to there and the fact that $\hat{v}$ is the unique maximizer of the profit with respect to the constant sell-down-to level) that $\hat{v} = \lim_{T\to\infty}\hat{v}_{1,T}$. Thus, to show that $\hat{v}$ is increasing in α, it suffices to show that $\hat{v}_{t,T}$ is increasing in α for every T and every $t \le T$. To show this, we use the analysis in Section 4.3.2 to conclude that

$$\hat{v}_{t,T} \quad = \quad \text{argmax}_{v\ge 0} - \overline{w}v - hv + \alpha\mathbb{E}[\hat{f}_{t+1,T}(v, Y_{t+1})],$$

$$\begin{aligned}
\text{where } \hat{f}_{t,T}(x,y) &= Sx \text{ if } y \geq x, \\
&= Sy + \overline{w}(x - y - \hat{v}_{t,T})^+ - h\min(x-y, \hat{v}_{t,T}) \\
&\quad + \alpha\mathbb{E}\left[\hat{f}_{t+1,T}(\min(x-y, \hat{v}_{t,T}), Y_{t+1})\right] \text{ if } y \leq x, \\
\text{and, } \hat{f}_{T+1,T}(x,y) &= 0 \text{ for all } x \geq 0 \text{ and } y \in \mathcal{Y}.
\end{aligned}$$

From the definition of $\hat{v}_{t,T}$ above and standard arguments justifying the interchange of the expectation and derivative operations, we know that

$$\alpha\mathbb{E}\left[\frac{\partial}{\partial v}\left(\hat{f}_{t+1,T}(v, Y_{t+1})\right)\Big| v = \hat{v}_{t,T}\right] = \overline{w} + h.$$

Observe that $\hat{f}_{t+1,T}(v,y)$ is concave in v for all $y \in \mathcal{Y}$ and that $\frac{\partial}{\partial v}\hat{f}_{t+1,T}(v,y) \geq 0$ for all $v \geq 0$ and for all $y \in \mathcal{Y}$ (Lemma 2). If we show that $\frac{\partial}{\partial x}\hat{f}_{t+1,T}(x,y)$ is increasing in α for all $x \geq 0$ and for all $y \in \mathcal{Y}$, then this will establish that $\hat{v}_{t,T}$ is increasing in α. Thus, we proceed to show inductively that $\frac{\partial}{\partial x}\hat{f}_{t,T}(x,y)$ is increasing in α for all (x,t,y).

Since $\hat{f}_{T+1,T}(x,y) = 0$ for all α, x and y, we assume that $\frac{\partial}{\partial x}\hat{f}_{t+1,T}(x,y)$ is increasing in α for all (x,y) and for some $t \leq T$. It only remains to use this to prove the claim that $\frac{\partial}{\partial x}\hat{f}_{t,T}(x,y)$ is also increasing in α for all (x,y).

From the definition of $\hat{f}_{t,T}(x,y)$ above, we see that when $y \geq x$, $\frac{\partial}{\partial x}\hat{f}_{t,T}(x,y) = S$ which is trivially increasing in α. When $y < x$, the proof of the claim is straightforward but is a cumbersome exercise whose main ideas we sketch and details we omit. Consider two discount factors $\alpha_2 > \alpha_1$. In what follows, we use the discount factor as an argument, where necessary. From the inductive assumption that $\frac{\partial}{\partial x}\hat{f}_{t+1,T}(x,y,\alpha)$ is increasing in α and the definition of $\hat{v}_{t,T}(\alpha)$, we see that $\hat{v}_{t,T}(\alpha_2) \geq \hat{v}_{t,T}(\alpha_1)$. Consider the three ranges, $x \in [y, y + \hat{v}_{t,T}(\alpha_1)]$, $x \in [y + \hat{v}_{t,T}(\alpha_1), y + \hat{v}_{t,T}(\alpha_2)]$ and $x \in [y + \hat{v}_{t,T}(\alpha_2), \infty)$. In each of these ranges, it can be shown that $\frac{\partial}{\partial x}\hat{f}_{t,T}(x,y,\alpha_2) \geq \frac{\partial}{\partial x}\hat{f}_{t,T}(x,y,\alpha_1)$. To demonstrate the inequality in the middle range, we use the following fact: From the definition of $\hat{v}_{t,T}$, we know that, for every $Y_{t+1} \in \mathcal{Y}$, the inequality $\alpha\mathbb{E}\left[\frac{\partial}{\partial x}\hat{f}_{t+1,T}(x-y, Y_{t+1})\right] \geq \overline{w} + h$ holds when $x - y < \hat{v}_{t,T}$. ■

Proof of Theorem 10: Similar to the proof of Theorem 9, we use the convergence of finite-horizon dynamic programs to infinite-horizon dynamic programs. Let $\hat{f}^A_{t,T}(x,y)$ and $\hat{f}^B_{t,T}(x,y)$ denote the optimal profit functions in Problem $\hat{P}_T$ corresponding to the sequences $\{Y^A_t\}$ and $\{Y^B_t\}$, respectively. Using standard DP convergence arguments, we know that $\hat{f}^A_\infty(Q,y) = \lim_{T\to\infty}\hat{f}^A_{1,T}(Q,y)$ and $\hat{f}^B_\infty(Q,y) = \lim_{T\to\infty}\hat{f}^B_{1,T}(Q,y)$. Then, to show the first result, it suffices to show the claim that, for all T, all $t \leq T$, and all (x,y), $\hat{f}^A_{t,T}(x,y) \geq$

$\hat{f}^B_{t,T}(x,y)$. Notice from the definitions of $\hat{f}^A_{t,T}(x,y)$ and $\hat{f}^B_{t,T}(x,y)$ that they can be written as follows: For all $t \leq T$,

$$\begin{aligned}
\hat{f}^A_{t,T}(x,y) = \max_{q^G,q^A} \quad & Sq^G + \overline{w}q^A - h(x-q^G-q^A) + \alpha\mathbb{E}\left[\hat{f}^A_{t+1,T}(x-q^G-q^A, Y^A_{t+1})\right] \\
\text{s.t.} \quad & q^A \geq 0, \quad 0 \leq q^G \leq y, \quad q^A + q^G \leq x \text{ , and} \\
\hat{f}^B_{t,T}(x,y) = \max_{q^G,q^A} \quad & Sq^G + \overline{w}q^A - h(x-q^G-q^A) + \alpha\mathbb{E}\left[\hat{f}^B_{t+1,T}(x-q^G-q^A, Y^B_{t+1})\right] \\
\text{s.t.} \quad & q^A \geq 0, \quad 0 \leq q^G \leq y, \quad q^A + q^G \leq x \text{ ,}
\end{aligned}$$

where $\hat{f}^A_{T+1,T}(x,y) = \hat{f}^B_{T+1,T}(x,y) = 0$ for all (x,y). Standard arguments involving the preservation of concavity under expectations, addition and maximization can be used to show that $\hat{f}^A_{t,T}(x,y)$ and $\hat{f}^B_{t,T}(x,y)$ are jointly concave in (x,y). Let us now assume inductively that $\hat{f}^A_{t+1,T}(x,y) \geq \hat{f}^B_{t+1,T}(x,y)$ for some $t \leq T$ and all (x,y), since this statement is trivially true for $t = T$. Then, we have

$$\mathbb{E}[\hat{f}^A_{t+1,T}(x-q^G-q^A, Y^A_{t+1})] \geq \mathbb{E}[\hat{f}^B_{t+1,T}(x-q^G-q^A, Y^A_{t+1})] \geq \mathbb{E}[\hat{f}^B_{t+1,T}(x-q^G-q^A, Y^B_{t+1})],$$

where the first inequality follows from the induction hypothesis and the second follows from the concavity of $\hat{f}^B_{t+1,T}$ with respect to its second argument and the assumption that $Y^A_{t+1} \leq_{cx} Y^B_{t+1}$. Thus, $\mathbb{E}[\hat{f}^A_{t+1,T}(x-q^G-q^A, Y^A_{t+1})] \geq \mathbb{E}[\hat{f}^B_{t+1,T}(x-q^G-q^A, Y^B_{t+1})]$; using this in the DP recursion above immediately implies the claim that $\hat{f}^A_{t,T}(x,y) \geq \hat{f}^B_{t,T}(x,y)$ for all (t,T,x,y). This completes the proof of the first result stated in the theorem.

We proceed to prove the second result. Since $\hat{f}^B_{t,T}(x,y)$ is jointly concave in (x,y), so is the limiting function $\hat{f}^B_\infty(x,y)$. We use the first result, the concavity of $\hat{f}^B_\infty(x,y)$ with respect to (x,y), and the convex ordering assumption as follows: $\mathbb{E}[\hat{f}^A_\infty(Q,Y^A_1)] \geq \mathbb{E}[\hat{f}^B_\infty(Q,Y^A_1)] \geq \mathbb{E}[\hat{f}^B_\infty(Q,Y^B_1)]$, which is the desired result. ■

Proof of Theorem 11: Assume that $Y_1 = y$ and $x_1 = Q$ in both P_T and $\hat{P}_\infty$. Assume further that $W_1 = w$ in P_T. Since $W_t \leq \overline{w}$ for every t, the expected total discounted profit of F in $\hat{P}_\infty$ is no smaller than the expected total discounted profit of F in P_T for every feasible policy. Thus, $\hat{f}_\infty(Q,y) \geq f_{1,T}(Q,y,w)$. We know from Theorem 8 that $\pi^{CS}(\hat{v})$ is the optimal policy in $\hat{P}_\infty$. Let $\hat{x}_t$, $\hat{q}^G_t$ and $\hat{q}^A_t$ denote the inventory at the beginning of period t, the quantity sold to G in period t and the quantity sold to A in period t, respectively, under this policy. Then, from the optimality of this policy in $\hat{P}_\infty$ and the feasibility of this policy in P_T, we have

$$\sum_{t=1}^{\infty} \alpha^{t-1}\mathbb{E}[S\hat{q}^G_t + \overline{w}\hat{q}^A_t - h\hat{x}_{t+1}] = \hat{f}_\infty(Q,y) \geq f_{1,T}(Q,y,w)$$

$$\geq \sum_{t=1}^{T} \alpha^{t-1} \mathbb{E}[S\hat{q}_t^G + W_t \hat{q}_t^A - h\hat{x}_{t+1}].$$

Therefore,

$$\begin{aligned}
\frac{\hat{f}_\infty(Q,y) - f_{1,T}(Q,y,w)}{f_{1,T}(Q,y,w)} \leq\ & \left(\frac{\overline{w}-w}{\underline{w}}\right) \frac{(Q-y-\hat{v})^+}{Q} \ + \\
& \frac{1}{\underline{w}Q} \sum_{t=T+1}^{\infty} \alpha^{t-1} \mathbb{E}[S\hat{q}_t^G + \overline{w}\hat{q}_t^A - h\hat{x}_{t+1}] \\
\leq\ & \frac{\overline{w}-\underline{w}}{\underline{w}} + \frac{1}{Q}\left(\frac{\alpha^T S}{\underline{w}}\right) \mathbb{E}\left[\left(\min\{(Q-y)^+, \hat{v}\} - \tilde{Y}_T\right)^+\right], \quad \text{(B.5)}
\end{aligned}$$

where the inequality in (B.5) is due to the facts that $w \geq \underline{w}$, $\hat{q}_t^A = 0$ for all $t \geq 2$, $\sum_{t=T+1}^{\infty} \alpha^{t-1}\hat{q}_t^G \leq \alpha^T \sum_{t=T+1}^{\infty} \hat{q}_t^G = \alpha^T(\min\{(Q-y)^+, \hat{v}\} - \tilde{Y}_T)^+$, and $h\hat{x}_{t+1} \geq 0$ for all t.

Consider the following cases:

- $Q \geq \hat{v}$: In this case, we have

$$\frac{1}{Q}\mathbb{E}\left[\left(\min\{(Q-y)^+, \hat{v}\} - \tilde{Y}_T\right)^+\right] \leq \frac{\mathbb{E}[(\hat{v} - \tilde{Y}_T)^+]}{\hat{v}}.$$

- $Q < \hat{v}$: In this case, we have

$$\begin{aligned}
&\frac{1}{Q}\mathbb{E}\left[\left(\min\{(Q-y)^+, \hat{v}\} - \tilde{Y}_T\right)^+\right] = \frac{1}{Q}\mathbb{E}\left[\left((Q-y)^+ - \tilde{Y}_T\right)^+\right] \\
&\leq \frac{1}{Q}\mathbb{E}[(Q-\tilde{Y}_T)^+] = \mathbb{E}[(1-\tilde{Y}_T/Q)^+] \leq \mathbb{E}[(1-\tilde{Y}_T/\hat{v})^+] = \frac{\mathbb{E}[(\hat{v}-\tilde{Y}_T)^+]}{\hat{v}}.
\end{aligned}$$

Next, we use the distribution-free newsvendor bounds of Gallego and Moon (1993) to obtain the following inequality:

$$\mathbb{E}\left[\left(\hat{v} - \tilde{Y}_T\right)^+\right] \leq \frac{\left[(T-1)\sigma^2 + (\hat{v} - (T-1)\mu)^2\right]^{1/2} + [\hat{v} - (T-1)\mu]}{2}.$$

The desired result follows by using this inequality and the above two cases in (B.5). ■

Proof of Theorem 12: For brevity, we avoid giving a detailed proof and only highlight the main ideas.

1. Recall from Section 4.4.2 that the threshold $\hat{v}$ is the solution to the following equation:

$$\sum_{t=2}^{\infty}(1-\alpha)\alpha^{t-1}P(\tilde{Y}_t > v) = \frac{\overline{w}(1-\alpha)+h}{S(1-\alpha)+h}. \tag{B.6}$$

Using the fact that the convolution of independently distributed exponential random variables follows an Erlang distribution, the left-hand-side of (B.6) can be expressed as

$$\sum_{t=2}^{\infty}(1-\alpha)\alpha^{t-1}p(\tilde{Y}_t > v) = \sum_{t=2}^{\infty}(1-\alpha)\alpha^{t-1}\sum_{n=0}^{t-2}\frac{(\lambda v)^n}{n!}e^{-\lambda v}.$$

Using standard algebraic manipulations, this equation becomes

$$\sum_{t=2}^{\infty}(1-\alpha)\alpha^{t-1}p(\tilde{Y}_t > v) = \alpha e^{-(1-\alpha)\lambda v}, \tag{B.7}$$

which, along with (B.6), gives (4.6).

2. Using the identity $A - \min(A, B) = (A - B)^+$, the expected amount of distressed sales is

$$\mathbb{E}_{Y_1}[(Q - Y_1 - \hat{v})^+] = (Q-\hat{v})^+ - \mathbb{E}_{Y_1}[\min\{(Q-\hat{v})^+, Y_1\}] = (Q-\hat{v})^+ - \frac{1-e^{-\lambda(Q-\hat{v})^+}}{\lambda}.$$

3. When $Q < \hat{v}$, the desired result in (4.7) follows directly from the definition of $\hat{G}(\cdot)$. Consider now the case $Q \geq \hat{v}$. Let $U(x) = -(\overline{w}+h)x + \alpha\hat{G}(x)\ \forall\ x$. Then, we have

$$\begin{aligned}
&\mathbb{E}_{Y_1}[\hat{f}_\infty(Q, Y_1)] \\
&= \mathbb{E}_{Y_1}[S\min\{Q, Y_1\} + (Q - Y_1 - \hat{v})^+\overline{w} - h\min\{(Q-Y_1)^+, \hat{v}\} + \\
&\qquad \alpha\hat{G}(\min\{(Q-Y_1)^+, \hat{v}\})] \\
&= \mathbb{E}_{Y_1}[S\min\{Q, Y_1\} + (Q-Y_1)^+\overline{w} + U(\min\{(Q-Y_1)^+, \hat{v}\})] \\
&= \frac{S}{\lambda}(1-e^{-\lambda Q}) + \overline{w}Q - \frac{\overline{w}}{\lambda}(1-e^{-\lambda Q}) + \mathbb{E}_{Y_1}\left[U(\min\{(Q-Y_1)^+, \hat{v}\})\right].
\end{aligned} \tag{B.8}$$

Recall from (B.4) that $\hat{G}(x) = [(1-\alpha)S+h]\sum_{t=2}^{\infty}\alpha^{t-2}\mathbb{E}[\min(x, \tilde{Y}_t)] - hx/(1-\alpha)$, which equals $[(1-\alpha)S+h]\int_0^x[\sum_{t=2}^{\infty}\alpha^{t-2}p(\tilde{Y}_t > z)]dz - hx/(1-\alpha)$. Using this and (B.7), we obtain the expression for $\hat{G}(x)$ in (4.9). Using (4.9), we obtain the closed-form expression for $\mathbb{E}_{Y_1}[U(\min\{(Q-Y_1)^+, \hat{v}\})]$, which together with (B.8), gives the desired result in (4.8). ■

Proof of Theorem 13: The proof follows directly by the definition of $L(Q)$, Theorem 10, and the fact that the exponential distribution with mean $1/\lambda$ is higher in the convex order than any NBUE distribution with the same mean (Theorem 3.A.55, Shaked and Shanthikumar 2007). ■

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